Section P.3

Functions and Their Graphs

- **Use function notation to represent and evaluate a function.**
- **Find the domain and range of a function.**
- **Sketch the graph of a function.**
- **Identify different types of transformations of functions.**
- **Classify functions and recognize combinations of functions.**

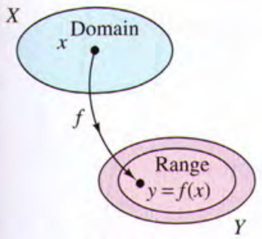
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$$A = \pi r^2 \qquad \text{A is a function of r.}$$

In this case r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable
Figure P.22

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure P.22).

Functions can be specified in a variety of ways. In this text, however, we will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$$x^2 + 2y = 1 \qquad \text{Equation in implicit form}$$

defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \frac{1}{2}(1 - x^2) \qquad \text{Equation in explicit form}$$

Using f as the name of the function, you can write this equation as

$$f(x) = \frac{1}{2}(1 - x^2). \qquad \text{Function notation}$$

The original equation, $x^2 + 2y = 1$, **implicitly** defines y as a function of x. When you solve the equation for y, you are writing the equation in **explicit** form.

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The symbol $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

FUNCTION NOTATION

The word *function* was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word "function" to describe any expression made up of a variable and some constants. He introduced the notation $y = f(x)$.

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For instance, the function given by

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\quad) = 2(\quad)^2 - 4(\quad) + 1$$

where parentheses are used instead of x. To evaluate $f(-2)$, simply place -2 in each set of parentheses.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 && \text{Substitute } -2 \text{ for } x. \\ &= 2(4) + 8 + 1 && \text{Simplify.} \\ &= 17 && \text{Simplify.} \end{aligned}$$

NOTE Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, the following equations all define the same function.

$$\begin{aligned} f(x) &= x^2 - 4x + 7 && \text{Function name is } f\text{, independent variable is } x. \\ f(t) &= t^2 - 4t + 7 && \text{Function name is } f\text{, independent variable is } t. \\ g(s) &= s^2 - 4s + 7 && \text{Function name is } g\text{, independent variable is } s. \end{aligned}$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate each expression.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0$

Solution

a. $$\begin{aligned} f(3a) &= (3a)^2 + 7 && \text{Substitute } 3a \text{ for } x. \\ &= 9a^2 + 7 && \text{Simplify.} \end{aligned}$$

b. $$\begin{aligned} f(b - 1) &= (b - 1)^2 + 7 && \text{Substitute } b - 1 \text{ for } x. \\ &= b^2 - 2b + 1 + 7 && \text{Expand binomial.} \\ &= b^2 - 2b + 8 && \text{Simplify.} \end{aligned}$$

c. $$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= 2x + \Delta x, \quad \Delta x \neq 0 \end{aligned}$$

STUDY TIP In calculus, it is important to communicate clearly the domain of a function or expression. For instance, in Example 1(c) the two expressions

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad 2x + \Delta x,$$

$$\Delta x \neq 0$$

are equivalent because $\Delta x = 0$ is excluded from the domain of each expression. Without a stated domain restriction, the two expressions would not be equivalent.

NOTE The expression in Example 1(c) is called a *difference quotient* and has a special significance in calculus. You will learn more about this in Chapter 2.

The Domain and Range of a Function

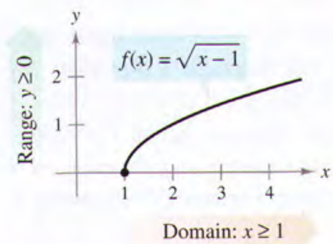
The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \qquad 4 \le x \le 5$$

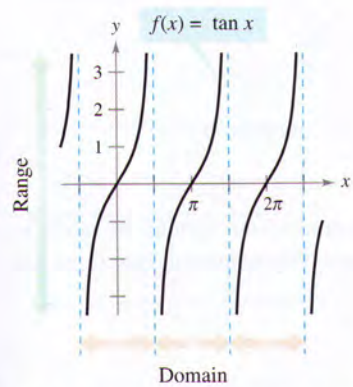
has an explicitly defined domain given by $\{x\colon 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x\colon x \neq \pm 2\}$.



(a) The domain of f is $[1, \infty)$ and the range is $[0, \infty)$.



(b) The domain of f is all x-values such that $x \neq \frac{\pi}{2} + n\pi$ and the range is $(-\infty, \infty)$.

Figure P.23

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range observe that $f(x) = \sqrt{x-1}$ is never negative. So, the range is the interval $[0, \infty)$, as indicated in Figure P.23(a).

b. The domain of the tangent function, as shown in Figure P.23(b),

$$f(x) = \tan x$$

is the set of all x-values such that

$$x \neq \frac{\pi}{2} + n\pi, \qquad n \text{ is an integer.} \qquad \text{Domain of tangent function}$$

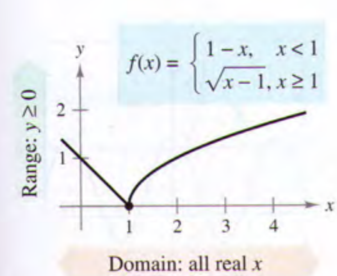
The range of this function is the set of all real numbers. For a review of the characteristics of this and other trigonometric functions, see Appendix D.

EXAMPLE 3 A Function Defined by More than One Equation

Determine the domain and range of the function.

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \sqrt{x - 1}, & \text{if } x \ge 1 \end{cases}$$

Solution Because f is defined for $x < 1$ and $x \ge 1$, the domain is the entire set of real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2(a). For $x < 1$, the values of $1 - x$ are positive. So, the range of the function is the interval $[0, \infty)$. (See Figure P.24.)

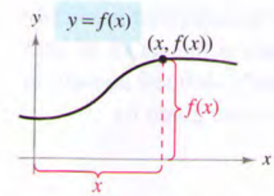


The domain of f is $(-\infty, \infty)$ and the range is $[0, \infty)$.

Figure P.24

A function from X to Y is **one-to-one** if to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function given in Example 2(a) is one-to-one, whereas the functions given in Examples 2(b) and 3 are not one-to-one. A function from X to Y is **onto** if its range consists of all of Y.

The Graph of a Function

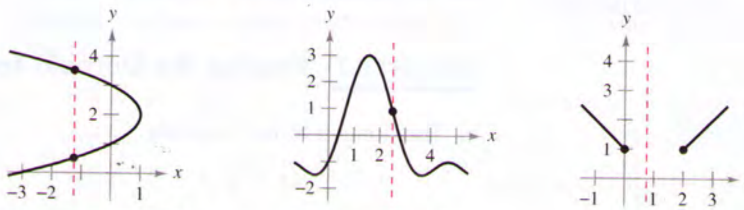


The graph of a function
Figure P.25

The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure P.25, note that

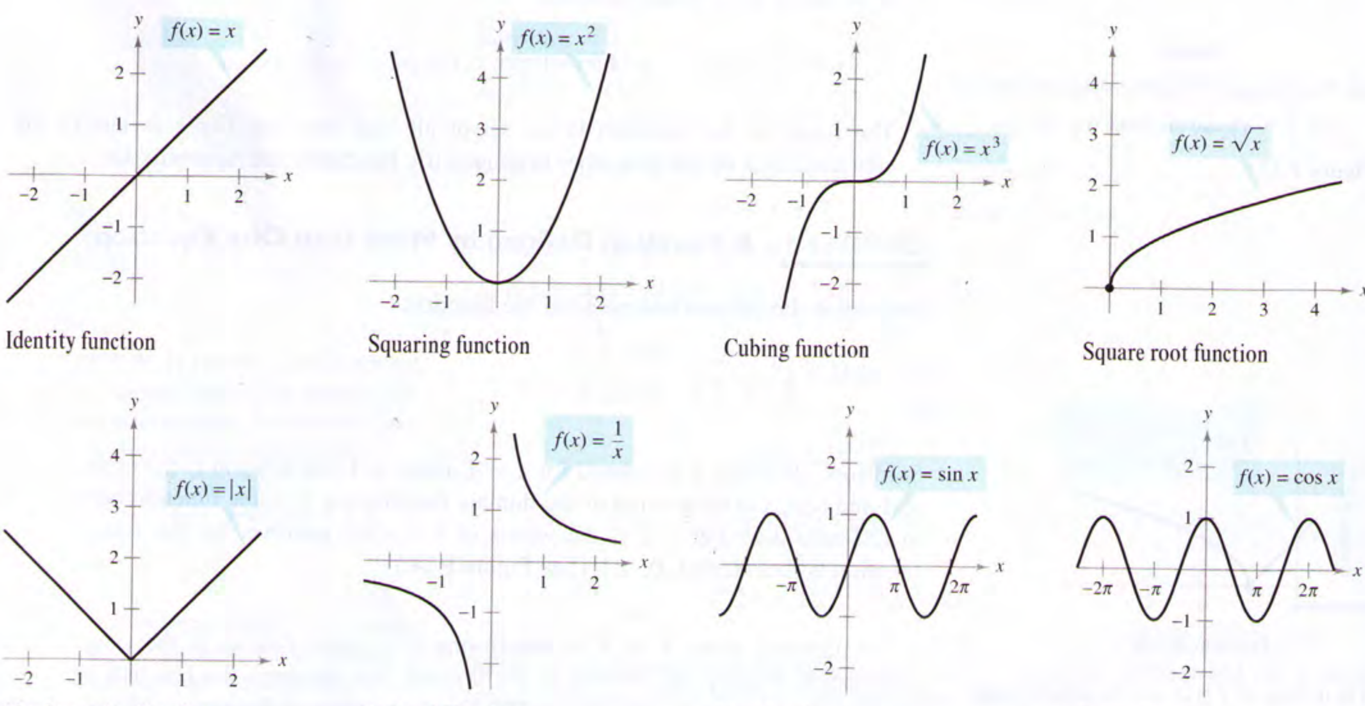
$$x = \text{the directed distance from the } y\text{-axis}$$
$$f(x) = \text{the directed distance from the } x\text{-axis.}$$

A vertical line can intersect the graph of a function of x at most *once*. This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x. That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point. For example, in Figure P.26(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice, whereas in Figures P.26(b) and (c), the graphs do define y as a function of x.



(a) Not a function of x **(b)** A function of x **(c)** A function of x
Figure P.26

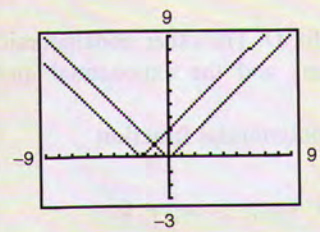
Figure P.27 shows the graphs of eight basic functions. You should be able to recognize these graphs. (Graphs of the other four basic trigonometric functions are shown in Appendix D.)



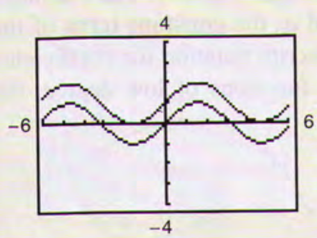
Identity function
Squaring function
Cubing function
Square root function
Absolute value function
Rational function
Sine function
Cosine function
The graphs of eight basic functions
Figure P.27

EXPLORATION

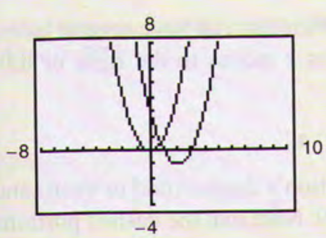
Writing Equations for Functions Each of the graphing utility screens below shows the graph of one of the eight basic functions shown on page 22. Each screen also shows a transformation of the graph. Describe the transformation. Then use your description to write an equation for the transformation.



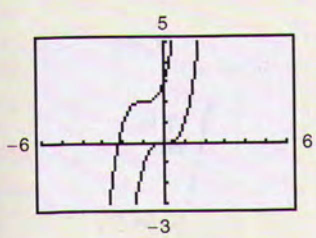
a.



b.



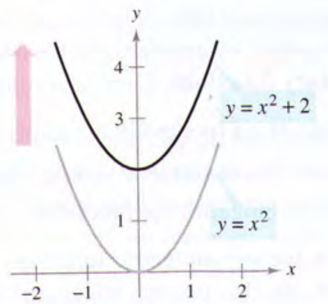
c.



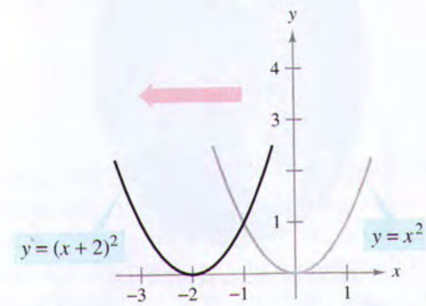
d.

Transformations of Functions

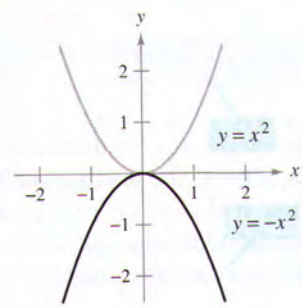
Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure P.28.



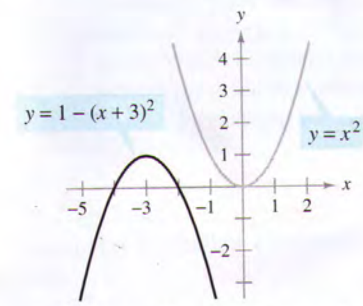
(a) Vertical shift upward



(b) Horizontal shift to the left



(c) Reflection



(d) Shift left, reflect, and shift upward

Figure P.28

Each of the graphs in Figure P.28 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane. For instance, if $f(x) = x^2$ is considered to be the original function in Figure P.28, the transformations shown can be represented by the following equations.

$y = f(x) + 2$	Vertical shift up 2 units
$y = f(x + 2)$	Horizontal shift to the left 2 units
$y = -f(x)$	Reflection about the x-axis
$y = -f(x + 3) + 1$	Shift left 3 units, reflect about x-axis, and shift up 1 unit

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right**:	$y = f(x - c)$
Horizontal shift c units to the **left**:	$y = f(x + c)$
Vertical shift c units **downward**:	$y = f(x) - c$
Vertical shift c units **upward**:	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Bettmann/Corbis

LEONHARD EULER (1707–1783)

In addition to making major contributions to almost every branch of mathematics, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics, and magnetism.

Classifications and Combinations of Functions

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler, to whom we are indebted for the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.** Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

You can review the trigonometric functions in Appendix D. The other nonalgebraic functions, such as the inverse trigonometric functions and the exponential and logarithmic functions, are introduced in Chapter 5.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0, \qquad a_n \neq 0$$

where the positive integer n is the **degree** of the polynomial function. The constants a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used.

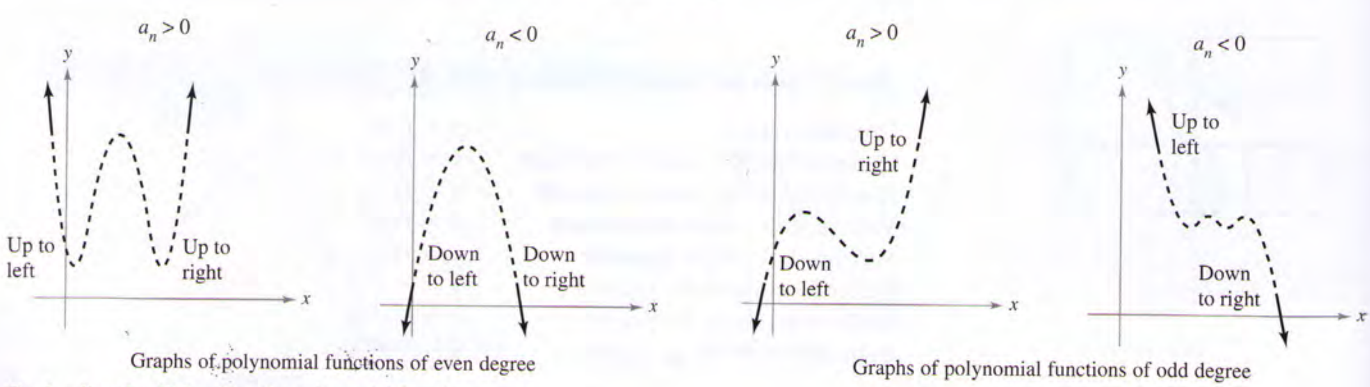
Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

FOR FURTHER INFORMATION For more on the history of the concept of a function, see the article "Evolution of the Function Concept: A Brief Survey" by Israel Kleiner in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure P.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



The Leading Coefficient Test for polynomial functions
Figure P.29

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

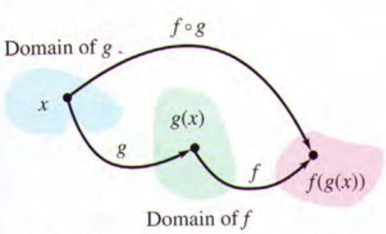
where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example, $f(x) = \sqrt{x+1}$ is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$$(f+g)(x) = f(x) + g(x) = (2x-3) + (x^2+1) \quad \text{Sum}$$
$$(f-g)(x) = f(x) - g(x) = (2x-3) - (x^2+1) \quad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = (2x-3)(x^2+1) \quad \text{Product}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x-3}{x^2+1} \quad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**



The domain of the composite function $f \circ g$
Figure P.30

Definition of Composite Function

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure P.30).

The composite of f with g may not be equal to the composite of g with f.

EXAMPLE 4 Finding Composite Functions

Given $f(x) = 2x - 3$ and $g(x) = \cos x$, find each composite function.

a. $f \circ g$ **b.** $g \circ f$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\cos x) && \text{Substitute } \cos x \text{ for } g(x).\\ &= 2(\cos x) - 3 && \text{Definition of } f(x)\\ &= 2\cos x - 3 && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(2x-3) && \text{Substitute } 2x-3 \text{ for } f(x).\\ &= \cos(2x-3) && \text{Definition of } g(x)\end{aligned}$$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.

EXPLORATION

Use a graphing utility to graph each function. Determine whether the function is *even*, *odd*, or *neither*.

$$f(x) = x^2 - x^4$$
$$g(x) = 2x^3 + 1$$
$$h(x) = x^5 - 2x^3 + x$$
$$j(x) = 2 - x^6 - x^8$$
$$k(x) = x^5 - 2x^4 + x - 2$$
$$p(x) = x^9 + 3x^5 - x^3 + x$$

Describe a way to identify a function as odd or even by inspecting the equation.

In Section P.1, an x-intercept of a graph was defined to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, the number a is a **zero** of f. In other words, *the zeros of a function f are the solutions of the equation* $f(x) = 0$. For example, the function $f(x) = x - 4$ has a zero at $x = 4$ because $f(4) = 0$.

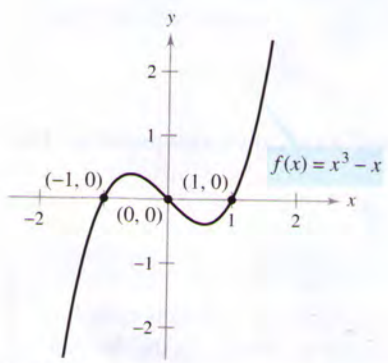
In Section P.1 you also studied different types of symmetry. In the terminology of functions, a function is **even** if its graph is symmetric with respect to the y-axis, and is **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section P.1 yield the following test for even and odd functions.

Test for Even and Odd Functions

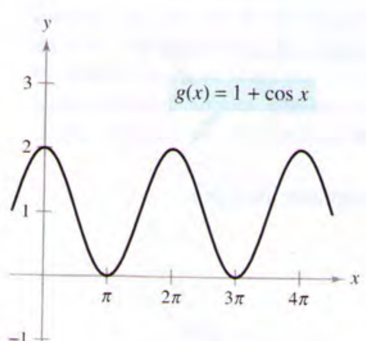
The function $y = f(x)$ is **even** if $f(-x) = f(x)$.

The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$.

NOTE Except for the constant function $f(x) = 0$, the graph of a function of x cannot have symmetry with respect to the x-axis because it then would fail the Vertical Line Test for the graph of the function.



(a) Odd function



(b) Even function

Figure P.31

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = 1 + \cos x$

Solution

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are found as shown.

$$x^3 - x = 0 \qquad \text{Let } f(x) = 0.$$
$$x(x^2 - 1) = x(x - 1)(x + 1) = 0 \qquad \text{Factor.}$$
$$x = 0, 1, -1 \qquad \text{Zeros of } f$$

See Figure P.31(a).

b. This function is even because

$$g(-x) = 1 + \cos(-x) = 1 + \cos x = g(x). \qquad \cos(-x) = \cos(x)$$

The zeros of g are found as shown.

$$1 + \cos x = 0 \qquad \text{Let } g(x) = 0.$$
$$\cos x = -1 \qquad \text{Subtract 1 from each side.}$$
$$x = (2n + 1)\pi,\ n \text{ is an integer.} \qquad \text{Zeros of } g$$

See Figure P.31(b).

NOTE Each of the functions in Example 5 is either even or odd. However, some functions, such as $f(x) = x^2 + x + 1$, are neither even nor odd.

Exercises for Section P.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graphs of f and g to answer the following.

(a) Identify the domains and ranges of f and g.

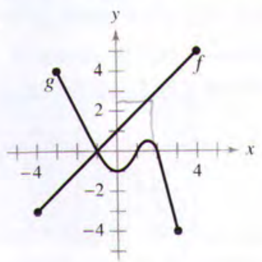
(b) Identify $f(-2)$ and $g(3)$.

(c) For what value(s) of x is $f(x) = g(x)$?

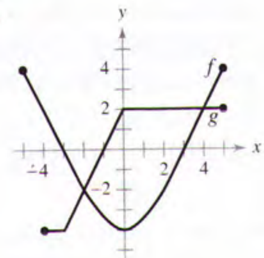
(d) Estimate the solution(s) of $f(x) = 2$.

(e) Estimate the solutions of $g(x) = 0$.

1.



2.



In Exercises 3–12, evaluate (if possible) the function at the given value(s) of the independent variable. Simplify the results.

3. $f(x) = 2x - 3$
(a) $f(0)$
(b) $f(-3)$
(c) $f(b)$
(d) $f(x - 1)$

4. $f(x) = \sqrt{x + 3}$
(a) $f(-2)$
(b) $f(6)$
(c) $f(-5)$
(d) $f(x + \Delta x)$

5. $g(x) = 3 - x^2$
(a) $g(0)$
(b) $g(\sqrt{3})$
(c) $g(-2)$
(d) $g(t - 1)$

6. $g(x) = x^2(x - 4)$
(a) $g(4)$
(b) $g(\frac{3}{2})$
(c) $g(c)$
(d) $g(t + 4)$

7. $f(x) = \cos 2x$
(a) $f(0)$
(b) $f(-\pi/4)$
(c) $f(\pi/3)$

8. $f(x) = \sin x$
(a) $f(\pi)$
(b) $f(5\pi/4)$
(c) $f(2\pi/3)$

9. $f(x) = x^3$
$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

10. $f(x) = 3x - 1$
$$\frac{f(x) - f(1)}{x - 1}$$

11. $f(x) = \dfrac{1}{\sqrt{x - 1}}$
$$\frac{f(x) - f(2)}{x - 2}$$

12. $f(x) = x^3 - x$
$$\frac{f(x) - f(1)}{x - 1}$$

In Exercises 13–18, find the domain and range of the function.

13. $h(x) = -\sqrt{x + 3}$

14. $g(x) = x^2 - 5$

15. $f(t) = \sec \dfrac{\pi t}{4}$

16. $h(t) = \cot t$

17. $f(x) = \dfrac{1}{x}$

18. $g(x) = \dfrac{2}{x - 1}$

In Exercises 19-24, find the domain of the function.

19. $f(x) = \sqrt{x} + \sqrt{1 - x}$

20. $f(x) = \sqrt{x^2 - 3x + 2}$

21. $g(x) = \dfrac{2}{1 - \cos x}$

22. $h(x) = \dfrac{1}{\sin x - \frac{1}{2}}$

23. $f(x) = \dfrac{1}{|x + 3|}$

24. $g(x) = \dfrac{1}{|x^2 - 4|}$

In Exercises 25–28, evaluate the function as indicated. Determine its domain and range.

25. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$ (d) $f(t^2 + 1)$

26. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$ (d) $f(s^2 + 2)$

27. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$

(a) $f(-3)$ (b) $f(1)$ (c) $f(3)$ (d) $f(b^2 + 1)$

28. $f(x) = \begin{cases} \sqrt{x + 4}, & x \leq 5 \\ (x - 5)^2, & x > 5 \end{cases}$

(a) $f(-3)$ (b) $f(0)$ (c) $f(5)$ (d) $f(10)$

In Exercises 29–36, sketch a graph of the function and find its domain and range. Use a graphing utility to verify your graph.

29. $f(x) = 4 - x$

30. $g(x) = \dfrac{4}{x}$

31. $h(x) = \sqrt{x - 1}$

32. $f(x) = \frac{1}{2}x^3 + 2$

33. $f(x) = \sqrt{9 - x^2}$

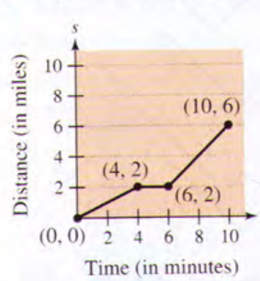
34. $f(x) = x + \sqrt{4 - x^2}$

35. $g(t) = 2 \sin \pi t$

36. $h(\theta) = -5 \cos \dfrac{\theta}{2}$

Writing About Concepts

37. The graph of the distance that a student drives in a 10-minute trip to school is shown in the figure. Give a verbal description of characteristics of the student's drive to school.

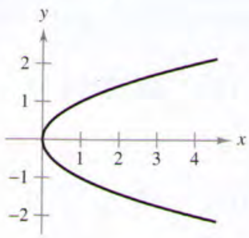


Writing About Concepts *(continued)*

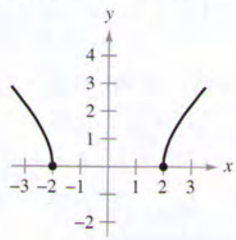
38. A student who commutes 27 miles to attend college remembers, after driving a few minutes, that a term paper that is due has been forgotten. Driving faster than usual, the student returns home, picks up the paper, and once again starts toward school. Sketch a possible graph of the student's distance from home as a function of time.

In Exercises 39–42, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

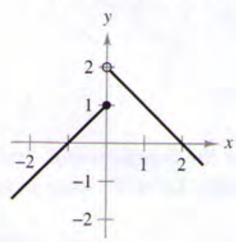
39. $x - y^2 = 0$



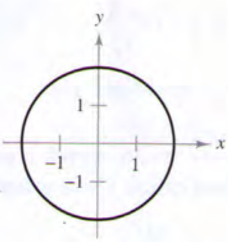
40. $\sqrt{x^2 - 4} - y = 0$



41. $y = \begin{cases} x + 1, & x \le 0 \\ -x + 2, & x > 0 \end{cases}$



42. $x^2 + y^2 = 4$



In Exercises 43–46, determine whether y is a function of x.

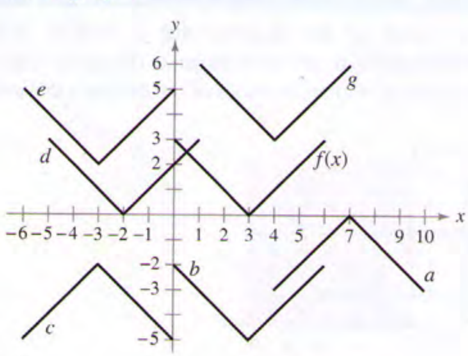
43. $x^2 + y^2 = 4$

44. $x^2 + y = 4$

45. $y^2 = x^2 - 1$

46. $x^2y - x^2 + 4y = 0$

In Exercises 47–52, use the graph of $y = f(x)$ to match the function with its graph.



47. $y = f(x + 5)$

48. $y = f(x) - 5$

49. $y = -f(-x) - 2$

50. $y = -f(x - 4)$

51. $y = f(x + 6) + 2$

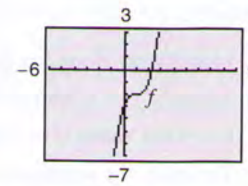
52. $y = f(x - 1) + 3$

53. Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x + 3)$ (b) $f(x - 1)$

(c) $f(x) + 2$ (d) $f(x) - 4$

(e) $3f(x)$ (f) $\frac{1}{4}f(x)$

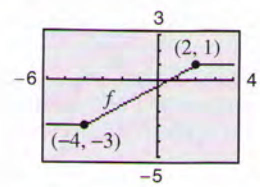


54. Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x - 4)$ (b) $f(x + 2)$

(c) $f(x) + 4$ (d) $f(x) - 1$

(e) $2f(x)$ (f) $\frac{1}{2}f(x)$



55. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of each function. In each case, describe the transformation.

(a) $y = \sqrt{x} + 2$ (b) $y = -\sqrt{x}$ (c) $y = \sqrt{x - 2}$

56. Specify a sequence of transformations that will yield each graph of h from the graph of the function $f(x) = \sin x$.

(a) $h(x) = \sin\left(x + \frac{\pi}{2}\right) + 1$ (b) $h(x) = -\sin(x - 1)$

57. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $g(f(0))$

(d) $f(g(-4))$ (e) $f(g(x))$ (f) $g(f(x))$

58. Given $f(x) = \sin x$ and $g(x) = \pi x$, evaluate each expression.

(a) $f(g(2))$ (b) $f\left(g\left(\frac{1}{2}\right)\right)$ (c) $g(f(0))$

(d) $g\left(f\left(\frac{\pi}{4}\right)\right)$ (e) $f(g(x))$ (f) $g(f(x))$

In Exercises 59–62, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the two composite functions equal?

59. $f(x) = x^2$

$g(x) = \sqrt{x}$

60. $f(x) = x^2 - 1$

$g(x) = \cos x$

61. $f(x) = \dfrac{3}{x}$

$g(x) = x^2 - 1$

62. $f(x) = \dfrac{1}{x}$

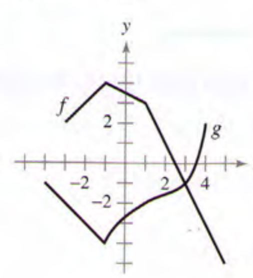
$g(x) = \sqrt{x + 2}$

63. Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$ (b) $g(f(2))$

(c) $g(f(5))$ (d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$ (f) $f(g(-1))$



64. ***Ripples*** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Think About It **In Exercises 65 and 66, $F(x) = f \circ g \circ h$. Identify functions for f, g, and h. (There are many correct answers.)**

65. $F(x) = \sqrt{2x - 2}$

66. $F(x) = -4 \sin(1 - x)$

In Exercises 67–70, determine whether the function is even, odd, or neither. Use a graphing utility to verify your result.

67. $f(x) = x^2(4 - x^2)$

68. $f(x) = \sqrt[3]{x}$

69. $f(x) = x \cos x$

70. $f(x) = \sin^2 x$

Think About It **In Exercises 71 and 72, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.**

71. $\left(-\frac{3}{2}, 4\right)$

72. $(4, 9)$

73. The graphs of f, g, and h are shown in the figure. Decide whether each function is even, odd, or neither.

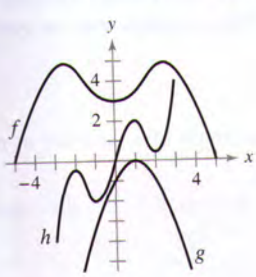


Figure for 73

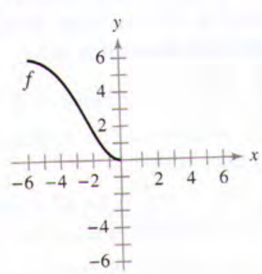


Figure for 74

74. The domain of the function f shown in the figure is $-6 \le x \le 6$.

(a) Complete the graph of f given that f is even.

(b) Complete the graph of f given that f is odd.

Writing Functions **In Exercises 75–78, write an equation for a function that has the given graph.**

75. Line segment connecting $(-4, 3)$ and $(0, -5)$

76. Line segment connecting $(1, 2)$ and $(5, 5)$

77. The bottom half of the parabola $x + y^2 = 0$

78. The bottom half of the circle $x^2 + y^2 = 4$

Modeling Data **In Exercises 79–82, match the data with a function from the following list.**

(i) $f(x) = cx$

(ii) $g(x) = cx^2$

(iii) $h(x) = c\sqrt{|x|}$

(iv) $r(x) = c/x$

Determine the value of the constant c for each function such that the function fits the data shown in the table.

79.

x	-4	-1	0	1	4
y	-32	-2	0	-2	-32

80.

x	-4	-1	0	1	4
y	-1	$-\frac{1}{4}$	0	$\frac{1}{4}$	1

81.

x	-4	-1	0	1	4
y	-8	-32	Undef.	32	8

82.

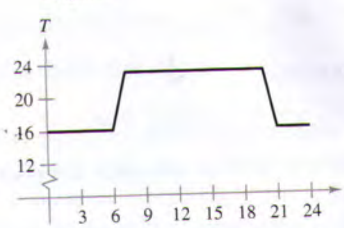
x	-4	-1	0	1	4
y	6	3	0	3	6

83. ***Graphical Reasoning*** An electronically controlled thermostat is programmed to lower the temperature during the night automatically (see figure). The temperature T in degrees Celsius is given in terms of t, the time in hours on a 24-hour clock.

(a) Approximate $T(4)$ and $T(15)$.

(b) The thermostat is reprogrammed to produce a temperature $H(t) = T(t - 1)$. How does this change the temperature? Explain.

(c) The thermostat is reprogrammed to produce a temperature $H(t) = T(t) - 1$. How does this change the temperature? Explain.

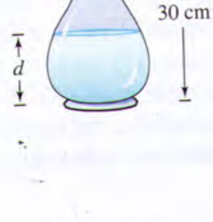


84. Water runs into a vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds. Use this information and the shape of the vase shown to answer the questions if d is the depth of the water in centimeters and t is the time in seconds (see figure).

(a) Explain why d is a function of t.

(b) Determine the domain and range of the function.

(c) Sketch a possible graph of the function.



85. ***Modeling Data*** The table shows the average numbers of acres per farm in the United States for selected years. *(Source: U.S. Department of Agriculture)*

Year	1950	1960	1970	1980	1990	2000
Acreage	213	297	374	426	460	434

(a) Plot the data where A is the acreage and t is the time in years, with $t = 0$ corresponding to 1950. Sketch a freehand curve that approximates the data.

(b) Use the curve in part (a) to approximate $A(15)$.

86. ***Automobile Aerodynamics*** The horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \qquad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph H.

(b) Rewrite the power function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.]

87. ***Think About It*** Write the function

$$f(x) = |x| + |x - 2|$$

without using absolute value signs. (For a review of absolute value, see Appendix D.)

88. ***Writing*** Use a graphing utility to graph the polynomial functions $p_1(x) = x^3 - x + 1$ and $p_2(x) = x^3 - x$. How many zeros does each function have? Is there a cubic polynomial that has no zeros? Explain.

89. Prove that the function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

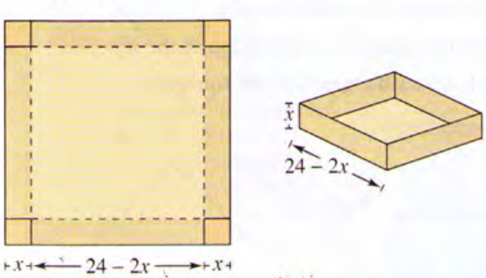
90. Prove that the function is even.

$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

91. Prove that the product of two even (or two odd) functions is even.

92. Prove that the product of an odd function and an even function is odd.

93. ***Volume*** An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).



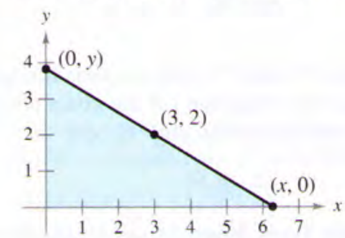
(a) Write the volume V as a function of x, the length of the corner squares. What is the domain of the function?

(b) Use a graphing utility to graph the volume function and approximate the dimensions of the box that yield a maximum volume.

(c) Use the *table* feature of a graphing utility to verify your answer in part (b). (The first two rows of the table are shown.)

Height, x	**Length and Width**	**Volume, V**
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

94. ***Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(3, 2)$ (see figure). Write the length L of the hypotenuse as a function of x.



True or False? **In Exercises 95–98, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If $f(a) = f(b)$, then $a = b$.

96. A vertical line can intersect the graph of a function at most once.

97. If $f(x) = f(-x)$ for all x in the domain of f, then the graph of f is symmetric with respect to the y-axis.

98. If f is a function, then $f(ax) = af(x)$.

Putnam Exam Challenge

99. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

100. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.

These problems were composed by the Committee on the Putnam Prize Competition.