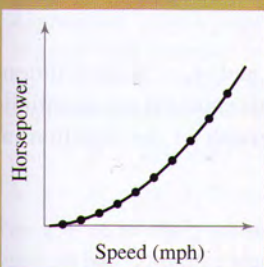
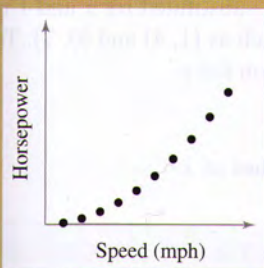
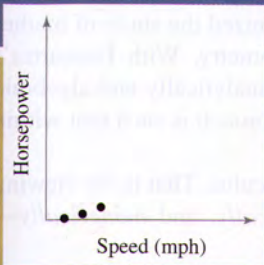


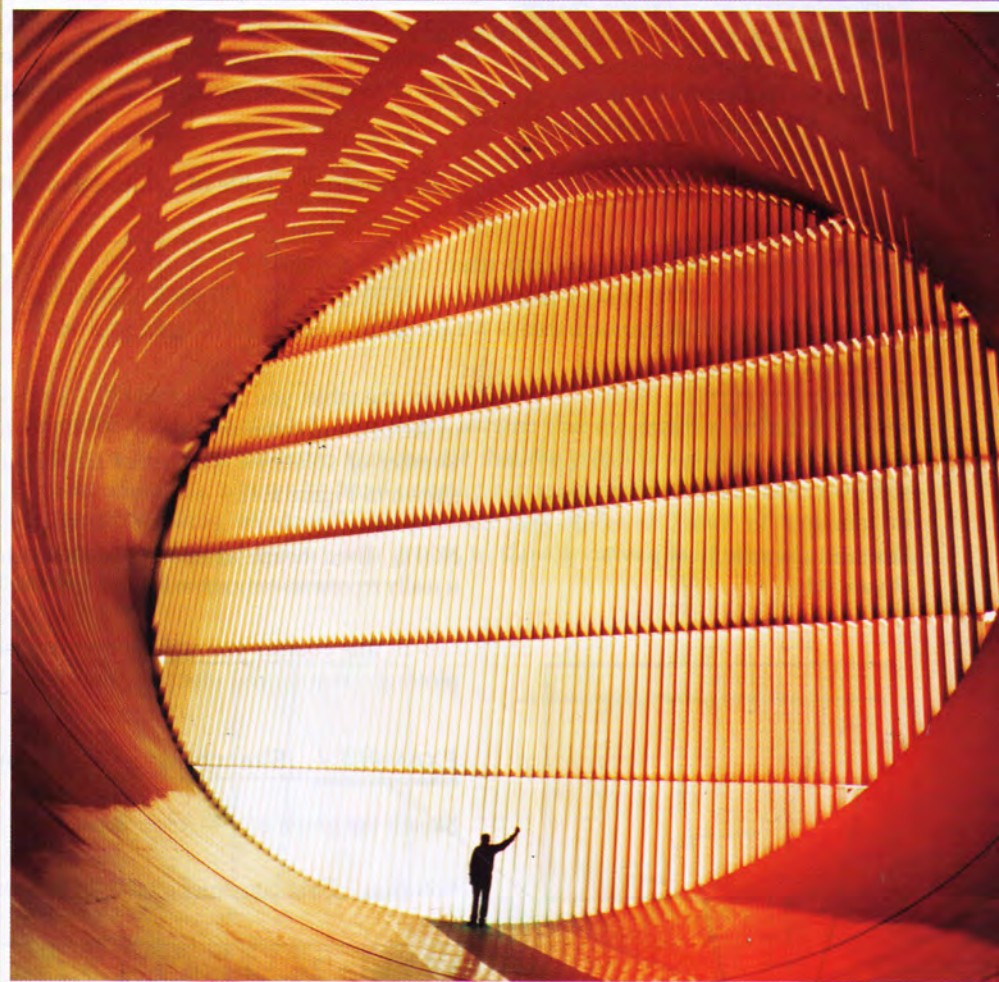
P

Preparation for Calculus



Two types of race cars designed and built by NASCAR teams are short track cars and super-speedway (long track) cars. The super-speedway race cars are subjected to extensive testing in wind tunnels like the one shown in the photo. Short track race cars and super-speedway race cars are designed either to allow for as much downforce as possible or to reduce the amount of drag on the race car. Which design do you think is used for each type of race car? Why?

Mathematical models are commonly used to describe data sets. These models can be represented by many different types of functions, such as linear, quadratic, cubic, rational, and trigonometric functions. In Chapter P, you will review how to find, graph, and compare mathematical models for different data sets.



Carol Anne Petrachenko/Corbis

Section P.1

Graphs and Models



Archive Photos

RENÉ DESCARTES (1596–1650)

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book *La Géométrie*, published in 1637.

- Sketch the graph of an equation.
- Find the intercepts of a graph.
- Test a graph for symmetry with respect to an axis and the origin.
- Find the points of intersection of two graphs.
- Interpret mathematical models for real-life data.

The Graph of an Equation

In 1637 the French mathematician René Descartes revolutionized the study of mathematics by joining its two major fields—algebra and geometry. With Descartes's coordinate plane, geometric concepts could be formulated analytically and algebraic concepts could be viewed graphically. The power of this approach is such that within a century, much of calculus had been developed.

The same approach can be followed in your study of calculus. That is, by viewing calculus from multiple perspectives—*graphically*, *analytically*, and *numerically*—you will increase your understanding of core concepts.

Consider the equation $3x + y = 7$. The point $(2, 1)$ is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y . This equation has many other solutions, such as $(1, 4)$ and $(0, 7)$. To find other solutions systematically, solve the original equation for y .

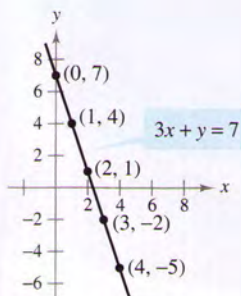
$$y = 7 - 3x$$

Analytic approach

Then construct a **table of values** by substituting several values of x .

x	0	1	2	3	4
y	7	4	1	-2	-5

Numerical approach



Graphical approach: $3x + y = 7$
Figure P.1

From the table, you can see that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solutions of the original equation $3x + y = 7$. Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure P.1.

NOTE Even though we refer to the sketch shown in Figure P.1 as the graph of $3x + y = 7$, it really represents only a *portion* of the graph. The entire graph would extend beyond the page.

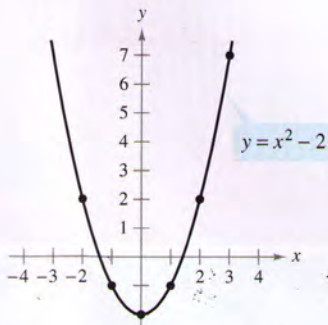
In this course, you will study many sketching techniques. The simplest is point plotting—that is, you plot points until the basic shape of the graph seems apparent.

EXAMPLE 1 Sketching a Graph by Point Plotting

Sketch the graph of $y = x^2 - 2$.

Solution First construct a table of values. Then plot the points shown in the table.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7



The parabola $y = x^2 - 2$
Figure P.2

Finally, connect the points with a *smooth curve*, as shown in Figure P.2. This graph is a **parabola**. It is one of the conics you will study in Chapter 10.

One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could badly misrepresent the graph. For instance, suppose that to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plotted only five points: $(-3, -3)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, and $(3, 3)$, as shown in Figure P.3(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure P.3(b).

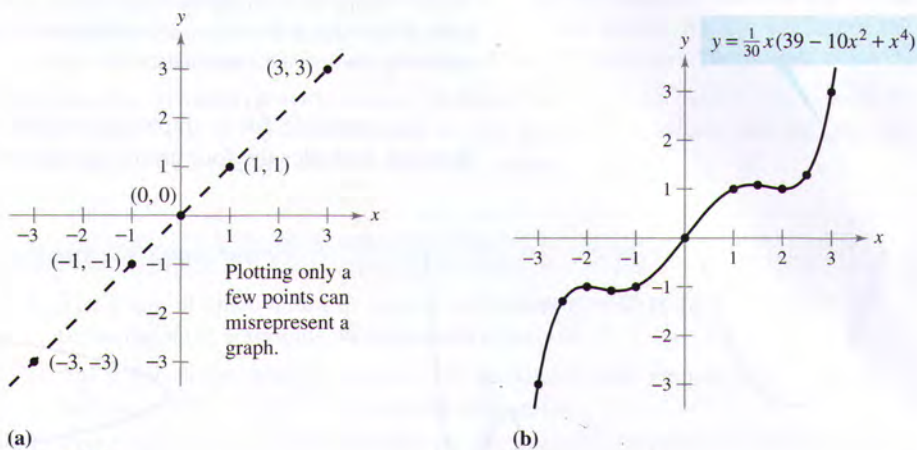


Figure P.3

EXPLORATION

Comparing Graphical and Analytic Approaches Use a graphing utility to graph each equation. In each case, find a viewing window that shows the important characteristics of the graph.

- $y = x^3 - 3x^2 + 2x + 5$
- $y = x^3 - 3x^2 + 2x + 25$
- $y = -x^3 - 3x^2 + 20x + 5$
- $y = 3x^3 - 40x^2 + 50x - 45$
- $y = -(x + 12)^3$
- $y = (x - 2)(x - 4)(x - 6)$

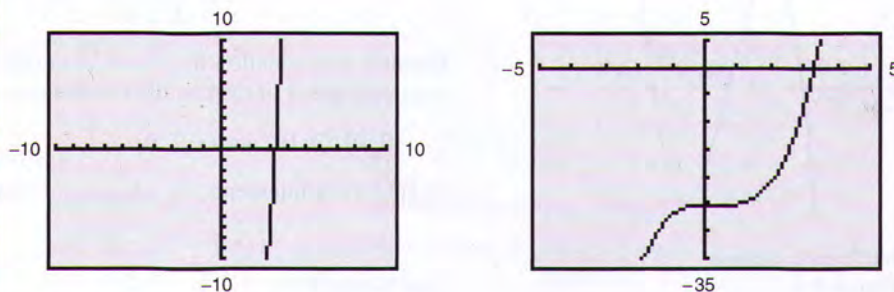
A purely graphical approach to this problem would involve a simple “guess, check, and revise” strategy. What types of things do you think an analytic approach might involve? For instance, does the graph have symmetry? Does the graph have turns? If so, where are they?

As you proceed through Chapters 1, 2, and 3 of this text, you will study many new analytic tools that will help you analyze graphs of equations such as these.

TECHNOLOGY Technology has made sketching of graphs easier. Even with technology, however, it is possible to misrepresent a graph badly. For instance, each of the graphing utility screens in Figure P.4 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. So, whether you are sketching a graph by hand or using a graphing utility, you must realize that different “viewing windows” can produce very different views of a graph. In choosing a viewing window, your goal is to show a view of the graph that fits well in the context of the problem.



Graphing utility screens of $y = x^3 - x^2 - 25$

Figure P.4

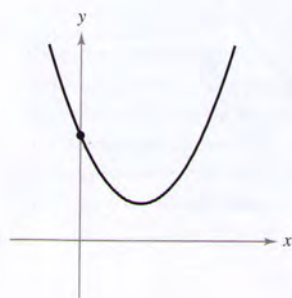
NOTE In this text, the term *graphing utility* means either a graphing calculator or computer graphing software such as *Maple*, *Mathematica*, *Derive*, *Mathcad*, or the *TI-89*.

Intercepts of a Graph

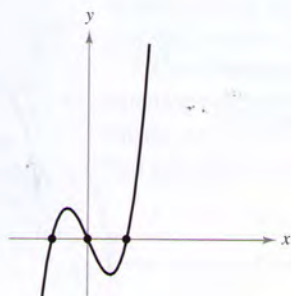
Two types of solution points that are especially useful in graphing an equation are those having zero as their x - or y -coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x - or y -axis. The point $(a, 0)$ is an **x -intercept** of the graph of an equation if it is a solution point of the equation. To find the x -intercepts of a graph, let y be zero and solve the equation for x . The point $(0, b)$ is a **y -intercept** of the graph of an equation if it is a solution point of the equation. To find the y -intercepts of a graph, let x be zero and solve the equation for y .

NOTE Some texts denote the x -intercept as the x -coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

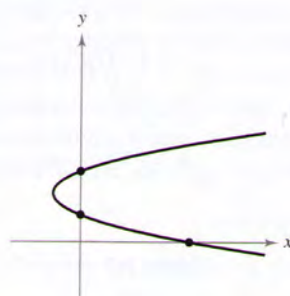
It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure P.5.



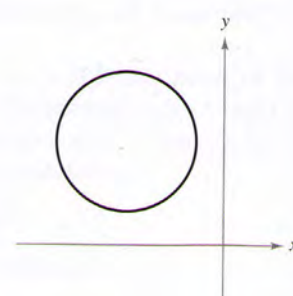
No x -intercepts
One y -intercept
Figure P.5



Three x -intercepts
One y -intercept



One x -intercept
Two y -intercepts



No intercepts

EXAMPLE 2 Finding x - and y -intercepts

Find the x - and y -intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x -intercepts, let y be zero and solve for x .

$$\begin{aligned} x^3 - 4x &= 0 && \text{Let } y \text{ be zero.} \\ x(x - 2)(x + 2) &= 0 && \text{Factor.} \\ x = 0, 2, \text{ or } -2 &&& \text{Solve for } x. \end{aligned}$$

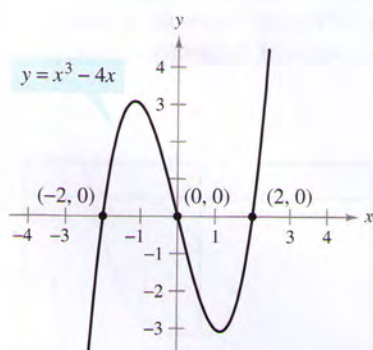
Because this equation has three solutions, you can conclude that the graph has three x -intercepts:

$$(0, 0), (2, 0), \text{ and } (-2, 0). \quad \text{\textit{x-intercepts}}$$

To find the y -intercepts, let x be zero. Doing this produces $y = 0$. So, the y -intercept is

$$(0, 0). \quad \text{\textit{y-intercept}}$$

(See Figure P.6.)



Intercepts of a graph
Figure P.6

TECHNOLOGY Example 2 uses an analytic approach to finding intercepts. When an analytic approach is not possible, you can use a graphical approach by finding the points at which the graph intersects the axes. Use a graphing utility to approximate the intercepts.

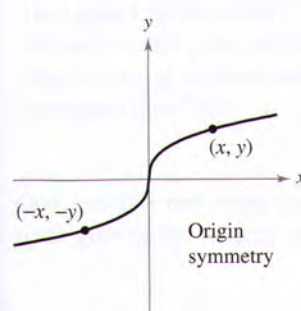
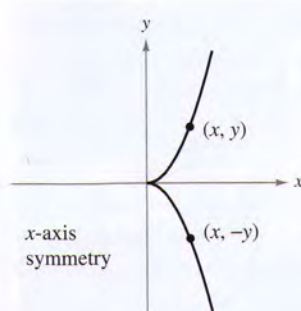
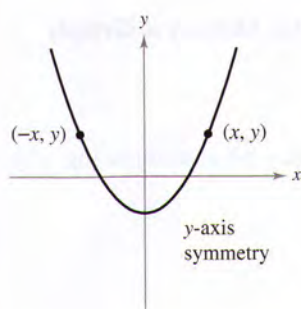


Figure P.7

Symmetry of a Graph

Knowing the symmetry of a graph *before* attempting to sketch it is useful because you need only half as many points to sketch the graph. The following three types of symmetry can be used to help sketch the graphs of equations (see Figure P.7).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph. This means that the portion of the graph above the x-axis is a mirror image of the portion below the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis if replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis if replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin if replacing x by $-x$ and y by $-y$ yields an equivalent equation.

The graph of a polynomial has symmetry with respect to the y-axis if each term has an even exponent (or is a constant). For instance, the graph of

$$y = 2x^4 - x^2 + 2 \quad \text{y-axis symmetry}$$

has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin if each term has an odd exponent, as illustrated in Example 3.

EXAMPLE 3 Testing for Origin Symmetry

Show that the graph of

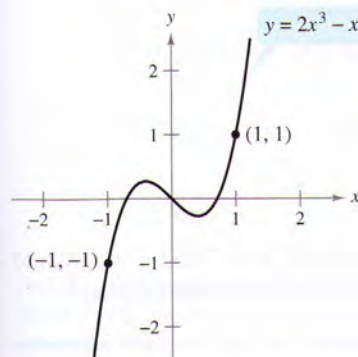
$$y = 2x^3 - x$$

is symmetric with respect to the origin.

Solution

$$\begin{array}{ll} y = 2x^3 - x & \text{Write original equation.} \\ -y = 2(-x)^3 - (-x) & \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y. \\ -y = -2x^3 + x & \text{Simplify.} \\ y = 2x^3 - x & \text{Equivalent equation} \end{array}$$

Because the replacements yield an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure P.8.



Origin symmetry
Figure P.8



EXAMPLE 4 Using Intercepts and Symmetry to Sketch a Graph

Sketch the graph of $x - y^2 = 1$.

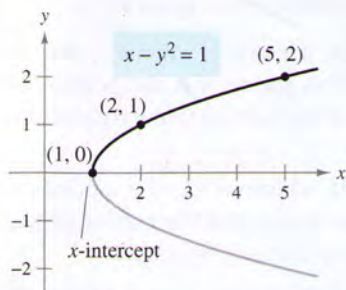


Figure P.9

Solution The graph is symmetric with respect to the x -axis because replacing y by $-y$ yields an equivalent equation.

$$\begin{array}{ll} x - y^2 = 1 & \text{Write original equation.} \\ x - (-y)^2 = 1 & \text{Replace } y \text{ by } -y. \\ x - y^2 = 1 & \text{Equivalent equation} \end{array}$$

This means that the portion of the graph below the x -axis is a mirror image of the portion above the x -axis. To sketch the graph, first plot the x -intercept and the points above the x -axis. Then reflect in the x -axis to obtain the entire graph, as shown in Figure P.9.

TECHNOLOGY Graphing utilities are designed so that they most easily graph equations in which y is a function of x (see Section P.3 for a definition of **function**). To graph other types of equations, you need to split the graph into two or more parts *or* you need to use a different graphing mode. For instance, to graph the equation in Example 4, you can split it into two parts.

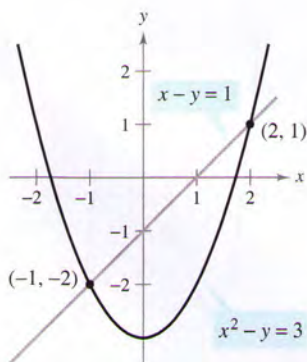
$$\begin{array}{ll} y_1 = \sqrt{x - 1} & \text{Top portion of graph} \\ y_2 = -\sqrt{x - 1} & \text{Bottom portion of graph} \end{array}$$

Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the points of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 Finding Points of Intersection

Find all points of intersection of the graphs of $x^2 - y = 3$ and $x - y = 1$.



Two points of intersection
Figure P.10

STUDY TIP You can check the points of intersection from Example 5 by substituting into *both* of the original equations or by using the *intersect* feature of a graphing utility.

Solution Begin by sketching the graphs of both equations on the *same* rectangular coordinate system, as shown in Figure P.10. Having done this, it appears that the graphs have two points of intersection. You can find these two points, as follows.

$$\begin{array}{ll} y = x^2 - 3 & \text{Solve first equation for } y. \\ y = x - 1 & \text{Solve second equation for } y. \\ x^2 - 3 = x - 1 & \text{Equate } y\text{-values.} \\ x^2 - x - 2 = 0 & \text{Write in general form.} \\ (x - 2)(x + 1) = 0 & \text{Factor.} \\ x = 2 \text{ or } -1 & \text{Solve for } x. \end{array}$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \text{ and } (-1, -2). \quad \text{Points of intersection}$$



indicates that in the HM mathSpace® CD-ROM and the online Eduspace® system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems Maple, Mathcad, Mathematica, and Derive.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models**. In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals: accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Section P.4 explores these goals more completely.

EXAMPLE 6 Comparing Two Mathematical Models

The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration y (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure P.11. In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035, using the quadratic model

$$y = 316.2 + 0.70t + 0.018t^2 \quad \text{Quadratic model for 1960–1990 data}$$

where $t = 0$ represents 1960, as shown in Figure P.11(a).

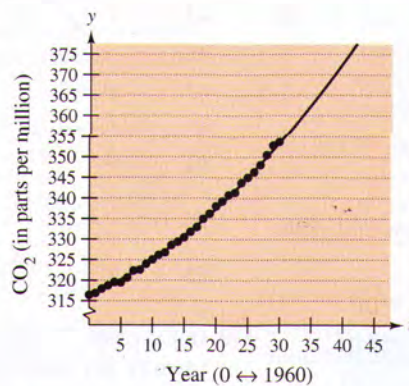
The data shown in Figure P.11(b) represent the years 1980 through 2002 and can be modeled by

$$y = 306.3 + 1.56t \quad \text{Linear model for 1980–2002 data}$$

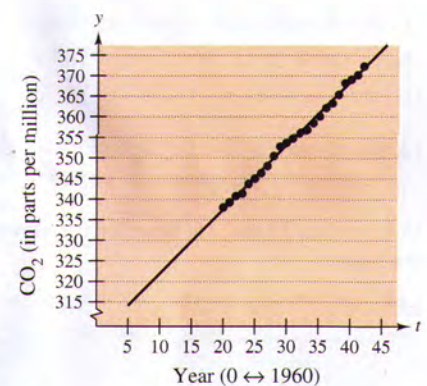
where $t = 0$ represents 1960. What was the prediction given in the *Scientific American* article in 1990? Given the new data for 1990 through 2002, does this prediction for the year 2035 seem accurate?



The Mauna Loa Observatory in Hawaii has been measuring the increasing concentration of carbon dioxide in Earth's atmosphere since 1958.



(a)



(b)

Figure P.11

Solution To answer the first question, substitute $t = 75$ (for 2035) into the quadratic model.

$$y = 316.2 + 0.70(75) + 0.018(75)^2 = 469.95 \quad \text{Quadratic model}$$

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035. Using the linear model for the 1980–2002 data, the prediction for the year 2035 is

$$y = 306.3 + 1.56(75) = 423.3 \quad \text{Linear model}$$

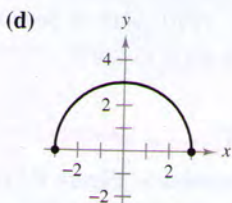
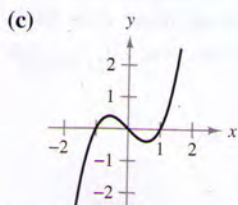
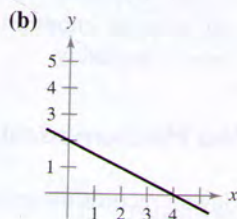
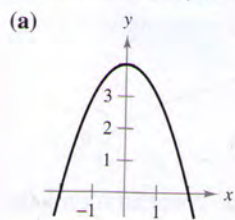
So, based on the linear model for 1980–2002, it appears that the 1990 prediction was too high.

NOTE The models in Example 6 were developed using a procedure called *least squares regression* (see Section 13.9). The quadratic and linear models have a correlation given by $r^2 = 0.997$ and $r^2 = 0.996$, respectively. The closer r^2 is to 1, the “better” the model.

Exercises for Section P.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the equation with its graph. [Graphs are labeled (a), (b), (c), and (d).]



1. $y = -\frac{1}{2}x + 2$
3. $y = 4 - x^2$

2. $y = \sqrt{9 - x^2}$
4. $y = x^3 - x$

In Exercises 5–14, sketch the graph of the equation by point plotting.

5. $y = \frac{3}{2}x + 1$

6. $y = 6 - 2x$

7. $y = 4 - x^2$

8. $y = (x - 3)^2$

9. $y = |x + 2|$

10. $y = |x| - 1$

11. $y = \sqrt{x} - 4$

12. $y = \sqrt{x + 2}$

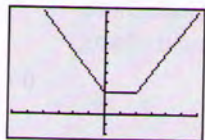
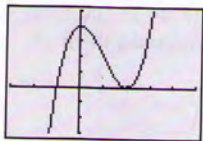
13. $y = \frac{2}{x}$

14. $y = \frac{1}{x - 1}$

In Exercises 15 and 16, describe the viewing window that yields the figure.

15. $y = x^3 - 3x^2 + 4$

16. $y = |x| + |x - 10|$



In Exercises 17 and 18, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places.

17. $y = \sqrt{5 - x}$

(a) $(2, y)$

(b) $(x, 3)$

18. $y = x^5 - 5x$

(a) $(-0.5, y)$

(b) $(x, -4)$

In Exercises 19–26, find any intercepts.

19. $y = x^2 + x - 2$

20. $y^2 = x^3 - 4x$

21. $y = x^2\sqrt{25 - x^2}$

22. $y = (x - 1)\sqrt{x^2 + 1}$

23. $y = \frac{3(2 - \sqrt{x})}{x}$

24. $y = \frac{x^2 + 3x}{(3x + 1)^2}$

25. $x^2y - x^2 + 4y = 0$

26. $y = 2x - \sqrt{x^2 + 1}$

In Exercises 27–38, test for symmetry with respect to each axis and to the origin.

27. $y = x^2 - 2$

28. $y = x^2 - x$

29. $y^2 = x^3 - 4x$

30. $y = x^3 + x$

31. $xy = 4$

32. $xy^2 = -10$

33. $y = 4 - \sqrt{x + 3}$

34. $xy - \sqrt{4 - x^2} = 0$

35. $y = \frac{x}{x^2 + 1}$

36. $y = \frac{x^2}{x^2 + 1}$

37. $y = |x^3 + x|$

38. $|y| - x = 3$

In Exercises 39–56, sketch the graph of the equation. Identify any intercepts and test for symmetry.

39. $y = -3x + 2$

40. $y = -\frac{1}{2}x + 2$

41. $y = \frac{1}{2}x - 4$

42. $y = \frac{2}{3}x + 1$

43. $y = 1 - x^2$

44. $y = x^2 + 3$

45. $y = (x + 3)^2$

46. $y = 2x^2 + x$

47. $y = x^3 + 2$

48. $y = x^3 - 4x$

49. $y = x\sqrt{x + 2}$

50. $y = \sqrt{9 - x^2}$

51. $x = y^3$

52. $x = y^2 - 4$

53. $y = \frac{1}{x}$

54. $y = \frac{10}{x^2 + 1}$

55. $y = 6 - |x|$

56. $y = |6 - x|$

In Exercises 57–60, use a graphing utility to graph the equation. Identify any intercepts and test for symmetry.

57. $y^2 - x = 9$

58. $x^2 + 4y^2 = 4$

59. $x + 3y^2 = 6$

60. $3x - 4y^2 = 8$

In Exercises 61–68, find the points of intersection of the graphs of the equations.

61. $x + y = 2$

62. $2x - 3y = 13$

$2x - y = 1$


$5x + 3y = 1$

63. $x^2 + y = 6$

64. $x = 3 - y^2$

$x + y = 4$

$y = x - 1$

The symbol  indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

$$65. \begin{aligned} x^2 + y^2 &= 5 \\ x - y &= 1 \end{aligned}$$

$$67. \begin{aligned} y &= x^3 \\ y &= x \end{aligned}$$

$$66. \begin{aligned} x^2 + y^2 &= 25 \\ 2x + y &= 10 \end{aligned}$$

$$68. \begin{aligned} y &= x^3 - 4x \\ y &= -(x + 2) \end{aligned}$$

Graphing Utility In Exercises 69–72, use a graphing utility to find the points of intersection of the graphs. Check your results analytically.

$$69. \begin{aligned} y &= x^3 - 2x^2 + x - 1 \\ y &= -x^2 + 3x - 1 \end{aligned}$$

$$70. \begin{aligned} y &= x^4 - 2x^2 + 1 \\ y &= 1 - x^2 \end{aligned}$$

$$71. \begin{aligned} y &= \sqrt{x+6} \\ y &= \sqrt{-x^2-4x} \end{aligned}$$

$$72. \begin{aligned} y &= -|2x-3| + 6 \\ y &= 6 - x \end{aligned}$$

Modeling Data The table shows the Consumer Price Index (CPI) for selected years. (Source: Bureau of Labor Statistics)

Year	1970	1975	1980	1985	1990	1995	2000
CPI	38.8	53.8	82.4	107.6	130.7	152.4	172.2

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the CPI and t represents the year, with $t = 0$ corresponding to 1970.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the CPI for the year 2010.

Modeling Data The table shows the average numbers of acres per farm in the United States for selected years. (Source: U.S. Department of Agriculture)

Year	1950	1960	1970	1980	1990	2000
Acreage	213	297	374	426	460	434

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the average acreage and t represents the year, with $t = 0$ corresponding to 1950.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the average number of acres per farm in the United States in the year 2010.

Break-Even Point Find the sales necessary to break even ($R = C$) if the cost C of producing x units is

$$C = 5.5\sqrt{x} + 10,000 \quad \text{Cost equation}$$

and the revenue R for selling x units is

$$R = 3.29x. \quad \text{Revenue equation}$$

Copper Wire The resistance y in ohms of 1000 feet of solid copper wire at 77°F can be approximated by the model

$$y = \frac{10,770}{x^2} - 0.37, \quad 5 \leq x \leq 100$$

where x is the diameter of the wire in mils (0.001 in.). Use a graphing utility to graph the model. If the diameter of the wire is doubled, the resistance is changed by about what factor?

Writing About Concepts

In Exercises 77 and 78, write an equation whose graph has the indicated property. (There may be more than one correct answer.)

77. The graph has intercepts at $x = -2$, $x = 4$, and $x = 6$.

78. The graph has intercepts at $x = -\frac{5}{2}$, $x = 2$, and $x = \frac{3}{2}$.

79. Each table shows solution points for one of the following equations.

(i) $y = kx + 5$

(ii) $y = x^2 + k$

(iii) $y = kx^{3/2}$

(iv) $xy = k$

Match each equation with the correct table and find k . Explain your reasoning.

(a)

x	1	4	9
y	3	24	81

(b)

x	1	4	9
y	7	13	23

(c)

x	1	4	9
y	36	9	4

(d)

x	1	4	9
y	-9	6	71

80. (a) Prove that if a graph is symmetric with respect to the x -axis and to the y -axis, then it is symmetric with respect to the origin. Give an example to show that the converse is not true.

(b) Prove that if a graph is symmetric with respect to one axis and to the origin, then it is symmetric with respect to the other axis.

True or False? In Exercises 81–84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

81. If $(1, -2)$ is a point on a graph that is symmetric with respect to the x -axis, then $(-1, -2)$ is also a point on the graph.

82. If $(1, -2)$ is a point on a graph that is symmetric with respect to the y -axis, then $(-1, -2)$ is also a point on the graph.

83. If $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has two x -intercepts.

84. If $b^2 - 4ac = 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has only one x -intercept.

In Exercises 85 and 86, find an equation of the graph that consists of all points (x, y) having the given distance from the origin. (For a review of the Distance Formula, see Appendix D.)

85. The distance from the origin is twice the distance from $(0, 3)$.

86. The distance from the origin is K ($K \neq 1$) times the distance from $(2, 0)$.