CHAPTER 7
Applications of Integration

Section 7.1 Area of a Region Between Two Curves

1. $A = \int_0^6 [0 - (x^2 - 6x)]\,dx = -\int_0^6 (x^2 - 6x)\,dx$

2. $A = \int_{-2}^{2} [(2x + 5) - (x^2 + 2x + 1)]\,dx$
$= \int_{-2}^{2} (-x^2 + 4)\,dx$

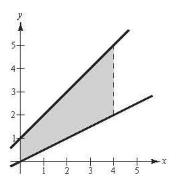
3. $A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)]\,dx$
$= \int_0^3 (-2x^2 + 6x)\,dx$

4. $A = \int_0^1 (x^2 - x^3)\,dx$

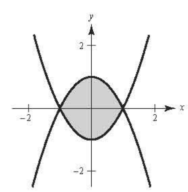
5. $A = 2\int_{-1}^{0} 3(x^3 - x)\,dx = 6\int_{-1}^{0} (x^3 - x)\,dx$
or $-6\int_0^1 (x^3 - x)\,dx$

6. $A = 2\int_0^1 [(x - 1)^3 - (x - 1)]\,dx$

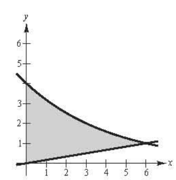
7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$



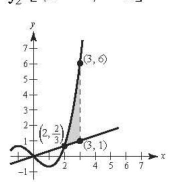
8. $\int_{-1}^{1} [(1 - x^2) - (x^2 - 1)]\,dx$



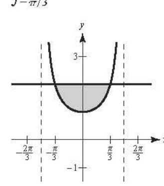
9. $\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6}\right] dx$



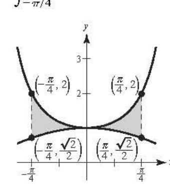
10. $\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$



11. $\int_{-\pi/3}^{\pi/3} (2 - \sec x)\,dx$

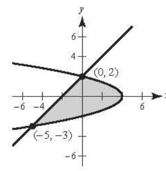


12. $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\,dx$



13. (a)

$$x = 4 - y^2$$
$$x = y - 2$$
$$4 - y^2 = y - 2$$
$$y^2 + y - 6 = 0$$
$$(y + 3)(y - 2) = 0$$



Intersection points: $(0, 2)$ and $(-5, -3)$

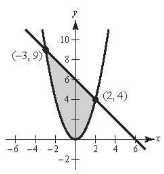
$$A = \int_{-5}^{0} \left[(x + 2) + \sqrt{4 - x}\right] dx + \int_{0}^{4} 2\sqrt{4 - x}\, dx = \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$$

(b) $A = \int_{-3}^{2} [(4 - y^2) - (y - 2)]\, dy = \frac{125}{6}$

14. (a) $y = x^2$ and $y = 6 - x$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$



$$A = \int_{-3}^{2} [(6 - x) - x^2]\, dx = \frac{125}{6}$$

(b) $A = \int_{0}^{4} 2\sqrt{y}\, dy + \int_{4}^{9} \left[(6 - y) + \sqrt{y}\right] dy$

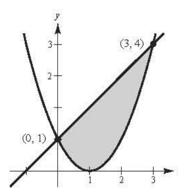
$= \frac{32}{3} + \frac{61}{6} = \frac{125}{6}$

15. $f(x) = x + 1$

$g(x) = (x - 1)^2$

$A \approx 4$

Matches (d)

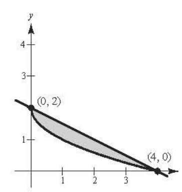


16. $f(x) = 2 - \frac{1}{2}x$

$g(x) = 2 - \sqrt{x}$

$A \approx 1$

Matches (a)

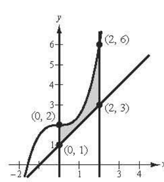


17. $A = \int_{0}^{2} \left[\left(\frac{1}{2}x^3 + 2\right) - (x + 1)\right] dx$

$= \int_{0}^{2} \left(\frac{1}{2}x^3 - x + 1\right) dx$

$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x\right]_{0}^{2}$

$= \left(\frac{16}{8} - \frac{4}{2} + 2\right) - 0 = 2$

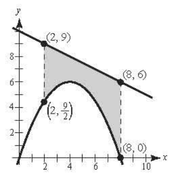


18. $A = \int_2^8 \left[\left(10 - \frac{1}{2}x\right) - \left(-\frac{3}{8}x(x-8)\right)\right] dx$

$= \int_2^8 \left(\frac{3}{8}x^2 - \frac{7}{2}x + 10\right) dx$

$= \left[\frac{x^3}{8} - \frac{7x^2}{4} + 10x\right]_2^8$

$= (64 - 112 + 80) - (1 - 7 + 20) = 18$



19. The points of intersection are given by:

$x^2 - 4x = 0$

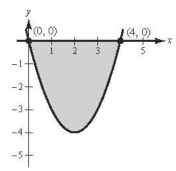
$x(x - 4) = 0$ when $x = 0, 4$

$A = \int_0^4 [g(x) - f(x)]\, dx$

$= -\int_0^4 (x^2 - 4x)\, dx$

$= -\left[\frac{x^3}{3} - 2x^2\right]_0^4$

$= \frac{32}{3}$



20. The points of intersection are given by:

$-x^2 + 4x + 1 = x + 1$

$-x^2 + 3x = 0$

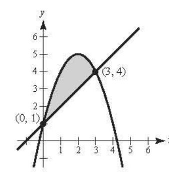
$x^2 = 3x$ when $x = 0, 3$

$A = \int_0^3 [(-x^2 + 4x + 1) - (x + 1)]\, dx$

$= \int_0^3 (-x^2 + 3x)\, dx$

$= \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3$

$= -9 + \frac{27}{2} = \frac{9}{2}$



21. The points of intersection are given by:

$x^2 + 2x + 1 = 3x + 3$

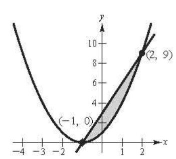
$(x - 2)(x + 1) = 0$ when $x = -1, 2$

$A = \int_{-1}^2 [g(x) - f(x)]\, dx$

$= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)]\, dx$

$= \int_{-1}^2 (2 + x - x^2)\, dx$

$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$



22. The points of intersection are given by:

$-x^2 + 4x + 2 = x + 2$

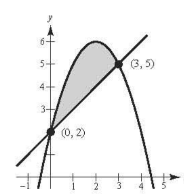
$x(3 - x) = 0$ when $x = 0, 3$

$A = \int_0^3 [f(x) - g(x)]\, dx$

$= \int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\, dx$

$= \int_0^3 (-x^2 + 3x)\, dx$

$= \left[\frac{-x^3}{3} + \frac{3}{2}x^2\right]_0^3 = \frac{9}{2}$



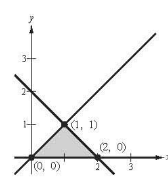
23. The points of intersection are given by:

$x = 2 - x$ and $x = 0$ and $2 - x = 0$

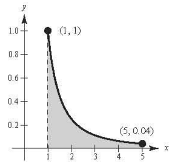
$x = 1$ $\quad x = 0$ $\quad x = 2$

$A = \int_0^1 [(2 - y) - (y)]\, dy = \left[2y - y^2\right]_0^1 = 1$

Note that if we integrate with respect to x, we need two integrals. Also, note that the region is a triangle.



24. $A = \int_1^5 \left(\frac{1}{x^2} - 0\right) dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$



25. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

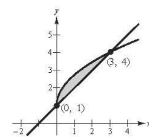
$$\sqrt{3x} = x \quad \text{when } x = 0, 3$$

$$A = \int_0^3 [f(x) - g(x)]\,dx$$

$$= \int_0^3 \left[\left(\sqrt{3x} + 1\right) - (x + 1)\right] dx$$

$$= \int_0^3 \left[(3x)^{1/2} - x\right] dx$$

$$= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2}$$



26. The points of intersection are given by:

$$\sqrt[3]{x - 1} = x - 1$$

$$x - 1 = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

$$x^3 - 3x^2 + 2x = 0$$

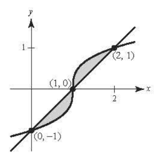
$$x(x^2 - 3x + 2) = 0$$

$$x(x - 2)(x - 1) = 0 \implies x = 0, 1, 2$$

$$A = 2\int_0^1 \left[(x - 1) - \sqrt[3]{x - 1}\right] dx$$

$$= 2\left[\frac{x^2}{2} - x - \frac{3}{4}(x - 1)^{4/3}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - 1 - 0\right) - \left(-\frac{3}{4}\right)\right] = \frac{1}{2}$$



27. The points of intersection are given by:

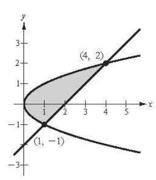
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$

$$A = \int_{-1}^2 [g(y) - f(y)]\,dy$$

$$= \int_{-1}^2 [(y + 2) - y^2]\,dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$$



28. The points of intersection are given by:

$$2y - y^2 = -y$$

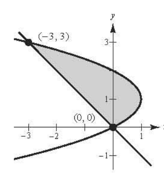
$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$

$$A = \int_0^3 [f(y) - g(y)]\,dy$$

$$= \int_0^3 [(2y - y^2) - (-y)]\,dy$$

$$= \int_0^3 (3y - y^2)\,dy$$

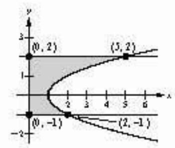
$$= \left[\frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^3 = \frac{9}{2}$$



29. $A = \int_{-1}^{2} [f(y) - g(y)]\, dy$

$= \int_{-1}^{2} [(y^2 + 1) - 0]\, dy$

$= \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6$

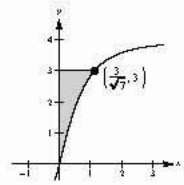


30. $A = \int_{0}^{3} [f(y) - g(y)]\, dy$

$= \int_{0}^{3} \left[\frac{y}{\sqrt{16 - y^2}} - 0\right] dy$

$= -\frac{1}{2}\int_{0}^{3} (16 - y^2)^{-1/2}(-2y)\, dy$

$= \left[-\sqrt{16 - y^2}\right]_{0}^{3} = 4 - \sqrt{7} \approx 1.354$



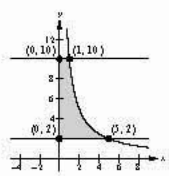
31. $y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$

$A = \int_{2}^{10} \frac{10}{y}\, dy$

$= \left[10 \ln y\right]_{2}^{10}$

$= 10(\ln 10 - \ln 2)$

$= 10 \ln 5 \approx 16.0944$

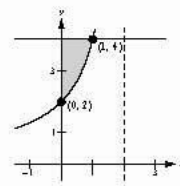


32. $A = \int_{0}^{1} \left(4 - \frac{4}{2 - x}\right) dx$

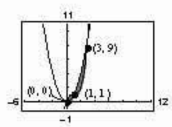
$= \left[4x + 4\ln|2 - x|\right]_{0}^{1}$

$= 4 - 4\ln 2$

≈ 1.227



33. (a)



(b) The points of intersection are given by:

$x^3 - 3x^2 + 3x = x^2$

$x(x - 1)(x - 3) = 0$ when $x = 0, 1, 3$

$A = \int_{0}^{1} [f(x) - g(x)]\, dx + \int_{1}^{3} [g(x) - f(x)]\, dx$

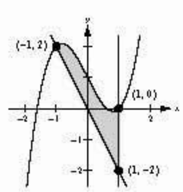
$= \int_{0}^{1} [(x^3 - 3x^2 + 3x) - x^2]\, dx + \int_{1}^{3} [x^2 - (x^3 - 3x^2 + 3x)]\, dx$

$= \int_{0}^{1} (x^3 - 4x^2 + 3x)\, dx + \int_{1}^{3} (-x^3 + 4x^2 - 3x)\, dx$

$= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_{0}^{1} + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_{1}^{3} = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$

(c) Numerical approximation: $0.417 + 2.667 \approx 3.083$

34. (a)



(b) The point of intersection is given by:

$x^3 - 2x + 1 = -2x$

$x^3 + 1 = 0$ when $x = -1$

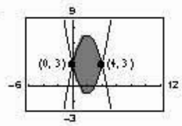
$A = \int_{-1}^{1} [f(x) - g(x)]\, dx$

$= \int_{-1}^{1} [(x^3 - 2x + 1) - (-2x)]\, dx$

$= \int_{-1}^{1} (x^3 + 1)\, dx = \left[\frac{x^4}{4} + x\right]_{-1}^{1} = 2$

(c) Numerical approximation: 2.0

35. (a)



(b) The points of intersection are given by:

$x^2 - 4x + 3 = 3 + 4x - x^2$

$2x(x - 4) = 0$ when $x = 0, 4$

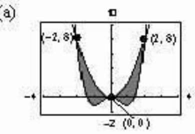
$$A = \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)]\,dx$$

$$= \int_0^4 (-2x^2 + 8x)\,dx$$

$$= \left[-\frac{2x^3}{3} + 4x^2\right]_0^4 = \frac{64}{3}$$

(c) Numerical approximation: 21.333

36. (a)



(b) The points of intersection are given by:

$x^4 - 2x^2 = 2x^2$

$x^2(x^2 - 4) = 0$ when $x = 0, \pm 2$

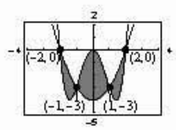
$$A = 2\int_0^2 [2x^2 - (x^4 - 2x^2)]\,dx$$

$$= 2\int_0^2 (4x^2 - x^4)\,dx$$

$$= 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{128}{15}$$

(c) Numerical approximation: 8.533

37. (a) $f(x) = x^4 - 4x^2,\ \ g(x) = x^2 - 4$



(b) The points of intersection are given by:

$x^4 - 4x^2 = x^2 - 4$

$x^4 - 5x^2 + 4 = 0$

$(x^2 - 4)(x^2 - 1) = 0$ when $x = \pm 2, \pm 1$

By symmetry:

$$A = 2\int_0^1 [(x^4 - 4x^2) - (x^2 - 4)]\,dx + 2\int_1^2 [(x^2 - 4) - (x^4 - 4x^2)]\,dx$$

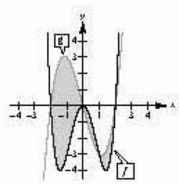
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\,dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$$

$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8$$

(c) Numerical approximation: $5.067 + 2.933 = 8.0$

38. (a) $f(x) = x^4 - 4x^2,\ \ g(x) = x^3 - 4x$



(b) The points of intersection are given by:

$x^4 - 4x^2 = x^3 - 4x$

$x^4 - x^3 - 4x^2 + 4x = 0$

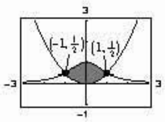
$x(x - 1)(x + 2)(x - 2) = 0$ when $x = -2, 0, 1, 2$

$$A = \int_{-2}^0 [(x^3 - 4x) - (x^4 - 4x^2)]\,dx + \int_0^1 [(x^4 - 4x^2) - (x^3 - 4x)]\,dx + \int_1^2 [(x^3 - 4x) - (x^4 - 4x^2)]\,dx$$

$$= \frac{248}{30} + \frac{37}{60} + \frac{53}{60} = \frac{293}{30}$$

(c) Numerical approximation: $8.267 + 0.617 + 0.883 \approx 9.767$

39. (a)



(b) The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x = \pm 1$$

$$A = 2\int_0^1 [f(x) - g(x)]\,dx$$

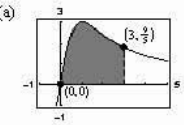
$$= 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2}\right]dx$$

$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

(c) Numerical approximation: 1.237

40. (a)



(b) $A = \int_0^3 \left[\frac{6x}{x^2+1} - 0\right]dx$

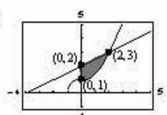
$$= \left[3\ln(x^2+1)\right]_0^3$$

$$= 3\ln 10$$

$$\approx 6.908$$

(c) Numerical approximation: 6.908

41. (a)

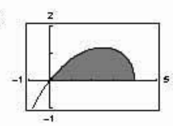


(b) and (c) $\sqrt{1+x^3} \le \frac{1}{2}x + 2$ on $[0, 2]$

You must use numerical integration because $y = \sqrt{1+x^3}$ does not have an elementary antiderivative.

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1+x^3}\right]dx \approx 1.759$$

42. (a)



(b) and (c) You must use numerical integration:

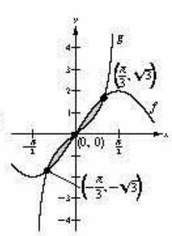
$$A = \int_0^4 x\sqrt{\frac{4-x}{4+x}}\,dx \approx 3.434$$

43. $A = 2\int_0^{\pi/3} [f(x) - g(x)]\,dx$

$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx$$

$$= 2\Big[-2\cos x + \ln|\cos x|\Big]_0^{\pi/3}$$

$$= 2(1 - \ln 2) \approx 0.614$$

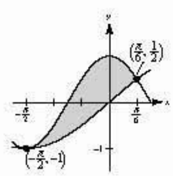


44. $A = \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x)\,dx$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\pi/2}^{\pi/6}$$

$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0)$$

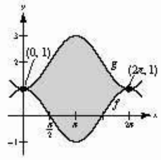
$$= \frac{3\sqrt{3}}{4} \approx 1.299$$



45. $A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\, dx$

$= 2\int_0^{2\pi} (1 - \cos x)\, dx$

$= 2\Big[x - \sin x\Big]_0^{2\pi} = 4\pi \approx 12.566$

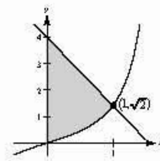


46. $A = \int_0^1 \left[\left(\sqrt{2} - 4\right)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$

$= \left[\frac{\sqrt{2} - 4}{2}x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4}\right]_0^1$

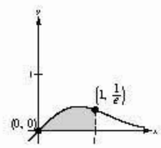
$= \left(\frac{\sqrt{2} - 4}{2} + 4 - \frac{4}{\pi}\sqrt{2}\right) - \left(-\frac{4}{\pi}\right)$

$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}\left(1 - \sqrt{2}\right) \approx 2.1797$



47. $A = \int_0^1 \left[xe^{-x^2} - 0\right] dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



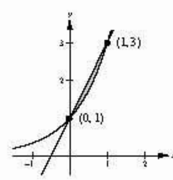
48. From the graph we see that f and g intersect twice at $x = 0$ and $x = 1$.

$A = \int_0^1 [g(x) - f(x)]\, dx$

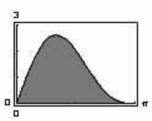
$= \int_0^1 [(2x + 1) - 3^x]\, dx$

$= \left[x^2 + x - \frac{1}{\ln 3}(3^x)\right]_0^1$

$= 2\left(1 - \frac{1}{\ln 3}\right) \approx 0.180$



49. (a)



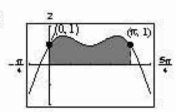
(b) $A = \int_0^{\pi} (2\sin x + \sin 2x)\, dx$

$= \left[-2\cos x - \frac{1}{2}\cos 2x\right]_0^{\pi}$

$= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) = 4$

(c) Numerical approximation: 4.0

50. (a)

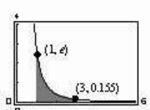


(b) $A = \int_0^{\pi} (2\sin x + \cos 2x)\, dx$

$= \left[-2\cos x + \frac{1}{2}\sin 2x\right]_0^{\pi} = 4$

(c) Numerical approximation: 4

51. (a)



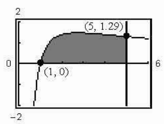
(b) $A = \int_1^3 \frac{1}{x^2}e^{1/x}\, dx$

$= \left[-e^{-1/x}\right]_1^3$

$= e - e^{1/3}$

(c) Numerical approximation: 1.323

52. (a)



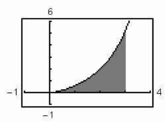
(b) $A = \int_1^5 \frac{4 \ln x}{x}\, dx$

$= \Big[2(\ln x)^2\Big]_1^5$

$= 2(\ln 5)^2$

(c) Numerical approximation: 5.181

53. (a)



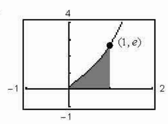
(b) The integral

$A = \int_0^3 \sqrt{\frac{x^3}{4 - x}}\, dx$

does not have an elementary antiderivative.

(c) $A \approx 4.7721$

54. (a)



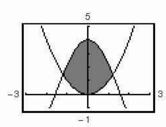
(b) The integral

$A = \int_0^1 \sqrt{x}\, e^x\, dx$

does not have an elementary antiderivative.

(c) 1.2556

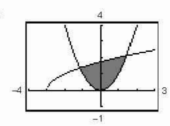
55. (a)



(b) The intersection points are difficult to determine by hand.

(c) Area $= \int_{-c}^{c} \left[4 \cos x - x^2\right] dx \approx 6.3043$ where $c \approx 1.201538$.

56. (a)



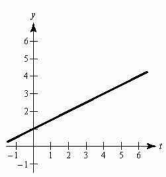
(b) The intersection points are difficult to determine.

(c) Intersection points: $(-1.164035, 1.3549778)$ and $(1.4526269, 2.1101248)$

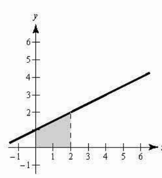
$A = \int_{-1.164035}^{1.4526269} \left[\sqrt{3 + x} - x^2\right] dx \approx 3.0578$

57. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

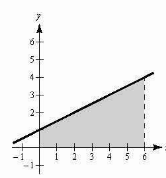
(a) $F(0) = 0$



(b) $F(2) = \frac{2^2}{4} + 2 = 3$

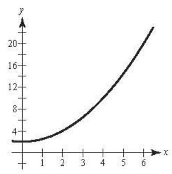


(c) $F(6) = \frac{6^2}{4} + 6 = 15$

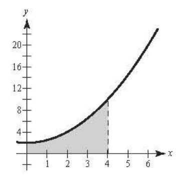


58. $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt = \left[\frac{1}{6}t^3 + 2t\right]_0^x = \frac{x^3}{6} + 2x$

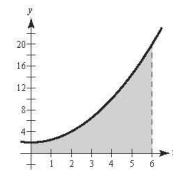
(a) $F(0) = 0$



(b) $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

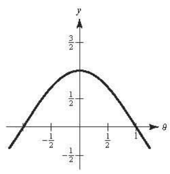


(c) $F(6) = 36 + 12 = 48$

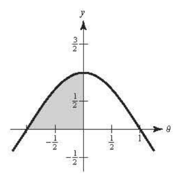


59. $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\, d\theta = \left[\frac{2}{\pi}\sin\frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2} + \frac{2}{\pi}$

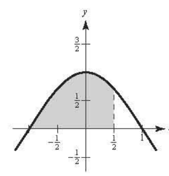
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$

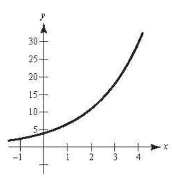


(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

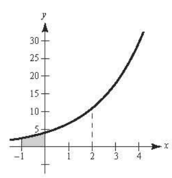


60. $F(y) = \int_{-1}^{y} 4e^{x/2}\, dx = \left[8e^{x/2}\right]_{-1}^{y} = 8e^{y/2} - 8e^{-1/2}$

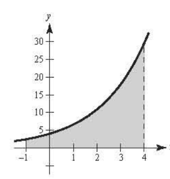
(a) $F(-1) = 0$



(b) $F(0) = 8 - 8e^{-1/2} \approx 3.1478$



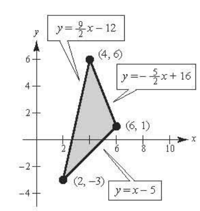
(c) $F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$



61. $A = \int_2^4 \left[\left(\frac{9}{2}x - 12\right) - (x - 5)\right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16\right) - (x - 5)\right] dx$

$= \int_2^4 \left(\frac{7}{2}x - 7\right) dx + \int_4^6 \left(-\frac{7}{2}x + 21\right) dx$

$= \left[\frac{7}{4}x^2 - 7x\right]_2^4 + \left[-\frac{7}{4}x^2 + 21x\right]_4^6 = 7 + 7 = 14$

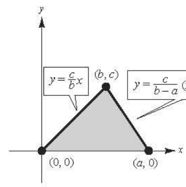


62. $A = \int_0^c \left[\left(\frac{b-a}{c}y + a\right) - \frac{b}{c}y\right] dy$

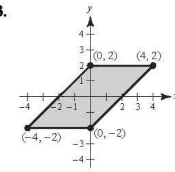
$= \int_0^c \left(-\frac{a}{c}y + a\right) dy$

$= \left[-\frac{a}{2c}y^2 + ay\right]_0^c$

$= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2}(\text{base})(\text{height})\right)$



63.



Left boundary line: $y = x + 2 \Leftrightarrow x = y - 2$

Right boundary line: $y = x - 2 \Leftrightarrow x = y + 2$

$A = \int_{-2}^{2} [(y + 2) - (y - 2)]\, dy$

$= \int_{-2}^{2} 4\, dy = 4y\Big]_{-2}^{2} = 8 - (-8) = 16$

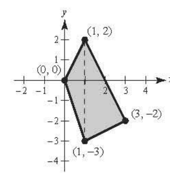
64. $A = \int_0^1 [2x - (-3x)]\, dx + \int_1^3 \left[(-2x + 4) - \left(\frac{1}{2}x - \frac{7}{2}\right)\right] dx$

$= \int_0^1 5x\, dx + \int_1^3 \left(-\frac{5}{2}x + \frac{15}{2}\right) dx$

$= \frac{5x^2}{2}\Big]_0^1 + \left[-\frac{5x^2}{4} + \frac{15}{2}x\right]_1^3$

$= \frac{5}{2} + \left[-\frac{45}{4} + \frac{45}{2} + \frac{5}{4} - \frac{15}{2}\right]$

$= \frac{15}{2}$



65. Answers will vary. If you let $\Delta x = 6$ and $n = 10$, $b - a = 10(6) = 60$.

(a) Area $\approx \frac{60}{2(10)}[0 + 2(14) + 2(14) + 2(12) + 2(12) + 2(15) + 2(20) + 2(23) + 2(25) + 2(26) + 0]$

$= 3[322] = 966$ sq ft

(b) Area $\approx \frac{60}{3(10)}[0 + 4(14) + 2(14) + 4(12) + 2(12) + 4(15) + 2(20) + 4(23) + 2(25) + 4(26) + 0]$

$= 2[502] = 1004$ sq ft

66. $\Delta x = 4, n = 8, b - a = (8)(4) = 32$

(a) Area $\approx \frac{32}{2(8)}[0 + 2(11) + 2(13.5) + 2(14.2) + 2(14) + 2(14.2) + 2(15) + 2(13.5) + 0]$

$= 2[190.8] = 381.6$ sq mi

(b) Area $\approx \frac{32}{3(8)}[0 + 4(11) + 2(13.5) + 4(14.2) + 2(14) + 4(14.2) + 2(15) + 4(13.5) + 0]$

$= \frac{4}{3}[296.6] \approx 395.5$ sq mi

67. $f(x) = x^3$

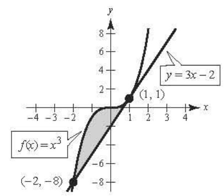
$f'(x) = 3x^2$

At $(1, 1)$, $f'(1) = 3$.

Tangent line: $y - 1 = 3(x - 1)$ or $y = 3x - 2$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^{1} [x^3 - (3x - 2)]\, dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \frac{27}{4}$$



68. $y = x^3 - 2x, \quad (-1, 1)$

$y' = 3x^2 - 2$

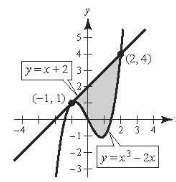
$y'(-1) = 3 - 2 = 1$

Tangent line: $y - 1 = 1(x + 1) \Rightarrow y = x + 2$

Intersection points: $(-1, 1)$ and $(2, 4)$

$$A = \int_{-1}^{2} [(x + 2) - (x^3 - 2x)]\, dx = \int_{-1}^{2} (-x^3 + 3x + 2)\, dx$$

$$= \left[-\frac{x^4}{4} + \frac{3x^2}{2} + 2x\right]_{-1}^{2} = \left[(-4 + 6 + 4) - \left(-\frac{1}{4} + \frac{3}{2} - 2\right)\right] = \frac{27}{4}$$



69. $f(x) = \dfrac{1}{x^2 + 1}$

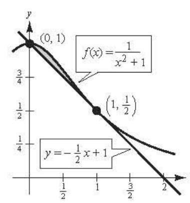
$f'(x) = -\dfrac{2x}{(x^2 + 1)^2}$

At $\left(1, \dfrac{1}{2}\right)$, $f'(1) = -\dfrac{1}{2}$.

Tangent line: $y - \dfrac{1}{2} = -\dfrac{1}{2}(x - 1)$ or $y = -\dfrac{1}{2}x + 1$

The tangent line intersects $f(x) = \dfrac{1}{x^2 + 1}$ at $x = 0$.

$$A = \int_{0}^{1} \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1\right)\right] dx = \left[\arctan x + \frac{x^2}{4} - x\right]_{0}^{1} = \frac{\pi - 3}{4} \approx 0.0354$$



70. $y = \dfrac{2}{1 + 4x^2}, \quad \left(\dfrac{1}{2}, 1\right)$

$y' = \dfrac{-16x}{(1 + 4x^2)^2}$

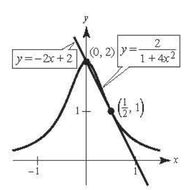
$y'\left(\dfrac{1}{2}\right) = \dfrac{-8}{2^2} = -2$

Tangent line: $y - 1 = -2\left(x - \dfrac{1}{2}\right)$

$y = -2x + 2$

Intersection points: $\left(\dfrac{1}{2}, 1\right), (0, 2)$

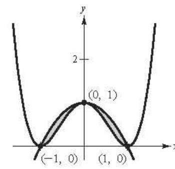
$$A = \int_{0}^{1/2} \left[\frac{2}{1 + 4x^2} - (-2x + 2)\right] dx = \left[\arctan(2x) + x^2 - 2x\right]_{0}^{1/2} = \arctan(1) + \frac{1}{4} - 1 = \frac{\pi}{4} - \frac{3}{4} \approx 0.0354$$



71. $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$A = \int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\, dx$$
$$= \int_{-1}^{1} (x^2 - x^4)\, dx$$
$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$$

You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.



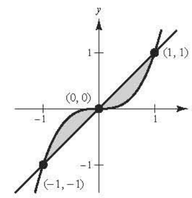
72. $x^3 \ge x$ on $[-1, 0]$, $x^3 \le x$ on $[0, 1]$

Both functions symmetric to origin.

$$\int_{-1}^{0} (x^3 - x)\, dx = -\int_{0}^{1} (x^3 - x)\, dx$$

Thus, $\int_{-1}^{1} (x^3 - x)\, dx = 0$.

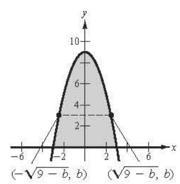
$$A = 2\int_{0}^{1} (x - x^3)\, dx$$
$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2}$$



73. Offer 2 is better because the accumulated salary (area under the curve) is larger.

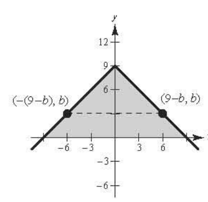
74. Proposal 2 is better since the cumulative deficit (the area under the curve) is less.

75.
$$A = \int_{-3}^{3} (9 - x^2)\, dx = 36$$
$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b]\, dx = 18$$
$$\int_{0}^{\sqrt{9-b}} [(9 - b) - x^2]\, dx = 9$$
$$\left[(9 - b)x - \frac{x^3}{3}\right]_{0}^{\sqrt{9-b}} = 9$$
$$\frac{2}{3}(9 - b)^{3/2} = 9$$
$$(9 - b)^{3/2} = \frac{27}{2}$$
$$9 - b = \frac{9}{\sqrt[3]{4}}$$
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



76. $A = 2\int_{0}^{9} (9 - x)\, dx = 2\left[9x - \frac{x^2}{2}\right]_{0}^{9} = 81$

$$2\int_{0}^{9-b} [(9 - x) - b]\, dx = \frac{81}{2}$$
$$2\int_{0}^{9-b} [(9 - b) - x]\, dx = \frac{81}{2}$$
$$2\left[(9 - b)x - \frac{x^2}{2}\right]_{0}^{9-b} = \frac{81}{2}$$
$$(9 - b)(9 - b) = \frac{81}{2}$$
$$9 - b = \frac{9}{\sqrt{2}}$$
$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



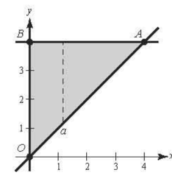
77. Area of triangle OAB is $\frac{1}{2}(4)(4) = 8$.

$$4 = \int_0^a (4 - x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^a = 4a - \frac{a^2}{2}$$

$$a^2 - 8a + 8 = 0$$

$$a = 4 \pm 2\sqrt{2}$$

Since $0 < a < 4$, select $a = 4 - 2\sqrt{2} \approx 1.172$.



78. Total area $= \int_{-2}^{2} (4 - y^2)\,dy = 2\int_0^2 (4 - y^2)\,dy$

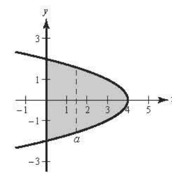
$$= 2\left[4y - \frac{y^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

$$\frac{16}{3} = 2\int_a^4 \sqrt{4 - x}\,dx = -\frac{4}{3}(4 - x)^{3/2}\Big]_a^4 = \frac{4}{3}(4 - a)^{3/2}$$

$$4 = (4 - a)^{3/2}$$

$$4^{2/3} = 4 - a$$

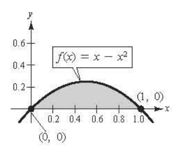
$$a = 4 - 4^{2/3} \approx 1.48$$



79. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\,\Delta x$

where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

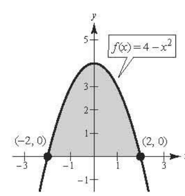
$$\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$



80. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\,\Delta x$

where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ is the same as

$$\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$



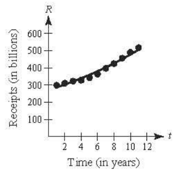
81. $\int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)]\,dt = \int_0^5 0.13t\,dt = \left[\frac{0.13t^2}{2}\right]_0^5 = \$1.625$ billion

82. $\int_0^5 [(7.21 + 0.26t + 0.02t^2) - (7.21 + 0.1t + 0.01t^2)]\,dt = \int_0^5 (0.01t^2 + 0.16t)\,dt$

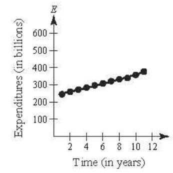
$$= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2}\right]_0^5$$

$= \frac{29}{12}$ billion $\approx \$2.417$ billion

83. (a) $y_1 = (270.3151)(1.0586)^t = 270.3151e^{0.05695t}$



(b) $y_2 = (239.9704)(1.0416)^t = 239.9704e^{0.04074t}$



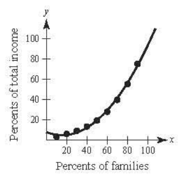
(c) Surplus $= \int_{12}^{17} (y_1 - y_2)\, dt \approx 926.4$ billion dollars

(Answers will vary.)

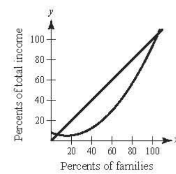
(d) No, $y_1 > y_2$ forever because $1.0586 > 1.0416$. No, these models are not accurate for the future. According to news, $E > R$ eventually.

84. (a) $y_1 = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) Income inequality $= \int_0^{100} [x - y_1]\, dx \approx 2006.7$

85. 5%: $P_1 = 893{,}000e^{(0.05)t}$

$3\frac{1}{2}$%: $P_2 = 893{,}000e^{(0.035)t}$

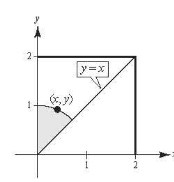
Difference in profits over 5 years: $\int_0^5 [893{,}000e^{0.05t} - 893{,}000e^{0.035t}]\, dt = 893{,}000\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5$

$\approx 893{,}000[(25.6805 - 34.0356) - (20 - 28.5714)]$

$\approx 893{,}000(0.2163) \approx \$193{,}156$

Note: Using a graphing utility, you obtain $193,183.

86. The total area is 8 times the area of the shaded region to the right. A point (x, y) is on the upper boundary of the region if



$$\sqrt{x^2 + y^2} = 2 - y$$
$$x^2 + y^2 = 4 - 4y + y^2$$
$$x^2 = 4 - 4y$$
$$4y = 4 - x^2$$
$$y = 1 - \frac{x^2}{4}.$$

We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$
$$x^2 + 4x - 4 = 0$$
$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \implies x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8\int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx = 8\left[x - \frac{x^3}{12} - \frac{x^2}{2}\right]_0^{-2+2\sqrt{2}} = \frac{16}{3}\left(4\sqrt{2} - 5\right) \approx 8(0.4379) = 3.503$$

87. The curves intersect at the point where the slope of y_2 equals that of y_1, 1.

$$y_2 = 0.08x^2 + k \Rightarrow y'_2 = 0.16x = 1 \Rightarrow x = \frac{1}{0.16} = 6.25$$

(a) The value of k is given by

$$y_1 = y_2$$
$$6.25 = (0.08)(6.25)^2 + k$$
$$k = 3.125.$$

(b) $$\text{Area} = 2\int_0^{6.25} (y_2 - y_1)\,dx = 2\int_0^{6.25} (0.08x^2 + 3.125 - x)\,dx = 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25} = 2(6.510417) \approx 13.02083$$

88. (a) $$A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5-x}\right)dx + \int_5^{5.5}(1-0)\,dx\right] = 2\left(\left[x + \frac{2}{9}(5-x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right) = 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\,V \approx 5000(12.062) = 60{,}310$ pounds

89. (a) $A \approx 6.031 - 2\left[\pi\left(\frac{1}{16}\right)^2\right] - 2\left[\pi\left(\frac{1}{8}\right)^2\right] \approx 5.908$

(b) $V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$

(c) $5000V \approx 5000(11.816) = 59{,}082$ pounds

90. True

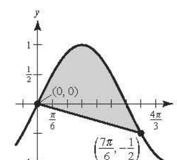
91. True

92. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)]\,dx = \int_0^2 [x - (2x - x^2)]\,dx = \frac{2}{3} \neq 0.$$

93. Line: $y = \frac{-3}{7\pi}x$

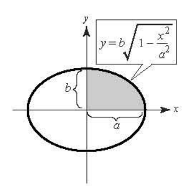
$$A = \int_0^{7\pi/6}\left[\sin x + \frac{3x}{7\pi}\right]dx = \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6} = \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1 \approx 2.7823$$



94. $$A = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$\int_0^a \sqrt{a^2 - x^2}\,dx$ is the area of $\frac{1}{4}$ of a circle $= \frac{\pi a^2}{4}$.

Hence, $A = \frac{4b}{a}\left(\frac{\pi a^2}{4}\right) = \pi ab.$



95. We want to find c such that:

$$\int_0^b [(2x - 3x^3) - c]\, dx = 0$$

$$\left[x^2 - \frac{3}{4}x^4 - cx\right]_0^b = 0$$

$$b^2 - \frac{3}{4}b^4 - cb = 0$$

But, $c = 2b - 3b^3$ because (b, c) is on the graph.

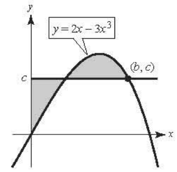
$$b^2 - \frac{3}{4}b^4 - (2b - 3b^3)b = 0$$

$$4 - 3b^2 - 8 + 12b^2 = 0$$

$$9b^2 = 4$$

$$b = \frac{2}{3}$$

$$c = \frac{4}{9}$$



Section 7.2 Volume: The Disk Method

1. $V = \pi \int_0^1 (-x + 1)^2\, dx = \pi \int_0^1 (x^2 - 2x + 1)\, dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}$

2. $V = \pi \int_0^2 (4 - x^2)^2\, dx = \pi \int_0^2 (x^4 - 8x^2 + 16)\, dx = \pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2 = \frac{256\pi}{15}$

3. $V = \pi \int_1^4 (\sqrt{x})^2\, dx = \pi \int_1^4 x\, dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

4. $V = \pi \int_0^3 \left(\sqrt{9 - x^2}\right)^2 dx = \pi \int_0^3 (9 - x^2)\, dx$

$$= \pi\left[9x - \frac{x^3}{3}\right]_0^3 = 18\pi$$

5. $V = \pi \int_0^1 [(x^2)^2 - (x^3)^2]\, dx = \pi \int_0^1 (x^4 - x^6)\, dx = \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{2\pi}{35}$

6. $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$$V = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4}\right)^2 - (2)^2\right] dx$$

$$= 2\pi \int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12\right] dx$$

$$= 2\pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x\right]_0^{2\sqrt{2}}$$

$$= 2\pi\left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2}\right]$$

$$= \frac{448\sqrt{2}}{15}\pi \approx 132.69$$

7. $y = x^2 \Rightarrow x = \sqrt{y}$

$$V = \pi \int_0^4 (\sqrt{y})^2\, dy = \pi \int_0^4 y\, dy$$

$$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$$

8. $y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2}$

$$V = \pi \int_0^4 \left(\sqrt{16 - y^2}\right)^2 dy = \pi \int_0^4 (16 - y^2)\, dy$$

$$= \pi\left[16y - \frac{y^3}{3}\right]_0^4 = \frac{128\pi}{3}$$

9. $y = x^{2/3} \Rightarrow x = y^{3/2}$

$$V = \pi \int_0^1 (y^{3/2})^2\, dy = \pi \int_0^1 y^3\, dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$$

10. $V = \pi\int_1^4 (-y^2 + 4y)^2\,dy = \pi\int_1^4 (y^4 - 8y^3 + 16y^2)\,dy$

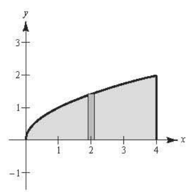
$= \pi\left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3}\right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5}$

11. $y = \sqrt{x},\ y = 0,\ x = 4$

(a) $R(x) = \sqrt{x},\ r(x) = 0$

$V = \pi\int_0^4 \left(\sqrt{x}\right)^2 dx$

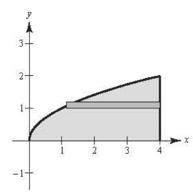
$= \pi\int_0^4 x\,dx = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$



(b) $R(y) = 4,\ r(y) = y^2$

$V = \pi\int_0^2 (16 - y^4)\,dy$

$= \pi\left[16y - \frac{1}{5}y^5\right]_0^2 = \frac{128\pi}{5}$

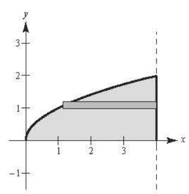


(c) $R(y) = 4 - y^2,\ r(y) = 0$

$V = \pi\int_0^2 (4 - y^2)^2\,dy$

$= \pi\int_0^2 (16 - 8y^2 + y^4)\,dy$

$= \pi\left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{256\pi}{15}$

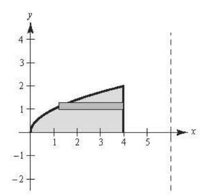


(d) $R(y) = 6 - y^2,\ r(y) = 2$

$V = \pi\int_0^2 \left[(6 - y^2)^2 - 4\right] dy$

$= \pi\int_0^2 (32 - 12y^2 + y^4)\,dy$

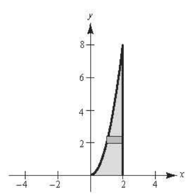
$= \pi\left[32y - 4y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{192\pi}{5}$



12. $y = 2x^2,\ y = 0,\ x = 2$

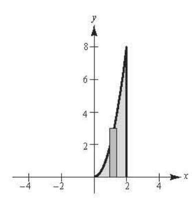
(a) $R(y) = 2,\ r(y) = \sqrt{y/2}$

$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right) dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$



(b) $R(x) = 2x^2,\ r(x) = 0$

$V = \pi\int_0^2 4x^4\,dx = \pi\left[\frac{4x^5}{5}\right]_0^2 = \frac{128\pi}{5}$



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12. —CONTINUED—

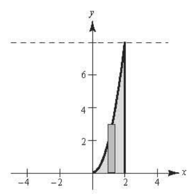
(c) $R(x) = 8,\ r(x) = 8 - 2x^2$

$$V = \pi\int_0^2 [64 - (64 - 32x^2 + 4x^4)]\,dx$$

$$= \pi\int_0^2 (32x^2 - 4x^4)\,dx = 4\pi\int_0^2 (8x^2 - x^4)\,dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$



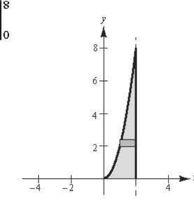
(d) $R(y) = 2 - \sqrt{y/2},\ r(y) = 0$

$$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8$$

$$= \frac{16\pi}{3}$$



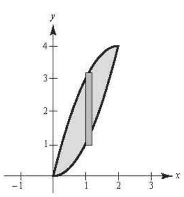
13. $y = x^2,\ y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2,\ r(x) = x^2$

$$V = \pi\int_0^2 [(4x - x^2)^2 - x^4]\,dx$$

$$= \pi\int_0^2 (16x^2 - 8x^3)\,dx$$

$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$

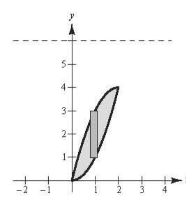


(b) $R(x) = 6 - x^2,\ r(x) = 6 - (4x - x^2)$

$$V = \pi\int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2]\,dx$$

$$= 8\pi\int_0^2 (x^3 - 5x^2 + 6x)\,dx$$

$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$



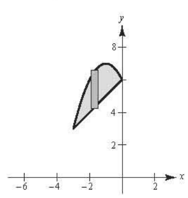
14. $y = 6 - 2x - x^2,\ y = x + 6$ intersect at $(-3, 3)$ and $(0, 6)$.

(a) $R(x) = 6 - 2x - x^2,\ r(x) = x + 6$

$$V = \pi\int_{-3}^0 [(6 - 2x - x^2)^2 - (x + 6)^2]\,dx$$

$$= \pi\int_{-3}^0 (x^4 + 4x^3 - 9x^2 - 36x)\,dx$$

$$= \pi\left[\frac{1}{5}x^5 + x^4 - 3x^3 - 18x^2\right]_{-3}^0 = \frac{243\pi}{5}$$

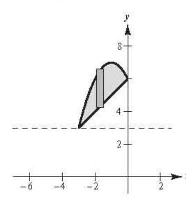


(b) $R(x) = (6 - 2x - x^2) - 3,\ r(x) = (x + 6) - 3$

$$V = \pi\int_{-3}^0 [(3 - 2x - x^2)^2 - (x + 3)^2]\,dx$$

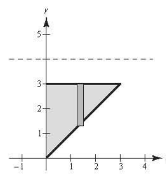
$$= \pi\int_{-3}^0 (x^4 + 4x^3 - 3x^2 - 18x)\,dx$$

$$= \pi\left[\frac{1}{5}x^5 + x^4 - x^3 - 9x^2\right]_{-3}^0 = \frac{108\pi}{5}$$



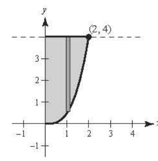
15. $R(x) = 4 - x, \; r(x) = 1$

$$V = \pi\int_0^3 [(4-x)^2 - (1)^2]\,dx$$
$$= \pi\int_0^3 (x^2 - 8x + 15)\,dx$$
$$= \pi\left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3$$
$$= 18\pi$$



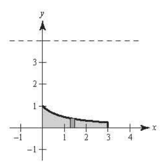
16. $R(x) = 4 - \frac{x^3}{2}, \; r(x) = 0$

$$V = \pi\int_0^1 \left(4 - \frac{x^3}{2}\right)^2 dx$$
$$= \pi\int_0^2 \left[16 - 4x^3 + \frac{x^6}{4}\right] dx$$
$$= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_0^2$$
$$= \pi\left[32 - 16 + \frac{128}{28}\right]$$
$$= \frac{144}{7}\pi$$



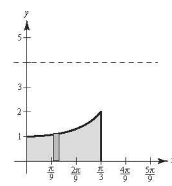
17. $R(x) = 4, \; r(x) = 4 - \frac{1}{1+x}$

$$V = \pi\int_0^3 \left[4^2 - \left(4 - \frac{1}{1+x}\right)^2\right] dx$$
$$= \pi\int_0^3 \left[\frac{8}{1+x} - \frac{1}{(1+x)^2}\right] dx$$
$$= \pi\left[8\ln(1+x) + \frac{1}{1+x}\right]_0^3$$
$$= \pi\left[8\ln 4 + \frac{1}{4} - 1\right]$$
$$= \left(8\ln 4 - \frac{3}{4}\right)\pi$$
$$\approx 32.485$$



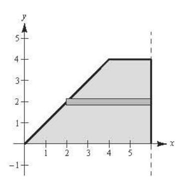
18. $R(x) = 4, \; r(x) = 4 - \sec x$

$$V = \pi\int_0^{\pi/3} [(4)^2 - (4 - \sec x)^2]\,dx$$
$$= \pi\int_0^{\pi/3} (8\sec x - \sec^2 x)\,dx$$
$$= \pi\Big[8\ln|\sec x + \tan x| - \tan x\Big]_0^{\pi/3}$$
$$= \pi\Big[\big(8\ln|2 + \sqrt{3}| - \sqrt{3}\big) - \big(8\ln|1 + 0| - 0\big)\Big]$$
$$= \pi\Big[8\ln\big(2 + \sqrt{3}\big) - \sqrt{3}\Big] \approx 27.66$$



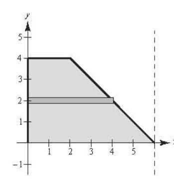
19. $R(y) = 6 - y, \; r(y) = 0$

$$V = \pi\int_0^4 (6-y)^2\,dy$$
$$= \pi\int_0^4 (y^2 - 12y + 36)\,dy$$
$$= \pi\left[\frac{y^3}{3} - 6y^2 + 36y\right]_0^4$$
$$= \frac{208\pi}{3}$$



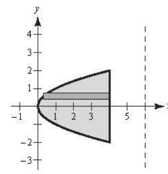
20. $R(y) = 6, \; r(y) = 6 - (6 - y) = y$

$$V = \pi\int_0^4 [(6)^2 - (y)^2]\,dy$$
$$= \pi\left[36y - \frac{y^3}{3}\right]_0^4 = \frac{368\pi}{3}$$



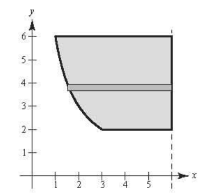
21. $R(y) = 6 - y^2,\ r(y) = 2$

$$V = \pi\int_{-2}^{2}\left[(6 - y^2)^2 - (2)^2\right] dy$$
$$= 2\pi\int_0^2 (y^4 - 12y^2 + 32)\, dy$$
$$= 2\pi\left[\frac{y^5}{5} - 4y^3 + 32y\right]_0^2$$
$$= \frac{384\pi}{5}$$



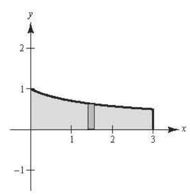
22. $R(y) = 6 - \dfrac{6}{y},\ r(y) = 0$

$$V = \pi\int_2^6 \left(6 - \frac{6}{y}\right)^2 dy$$
$$= 36\pi\int_2^6 \left(1 - \frac{2}{y} + \frac{1}{y^2}\right) dy$$
$$= 36\pi\left[y - 2\ln|y| - \frac{1}{y}\right]_2^6$$
$$= 36\pi\left[\left(\frac{35}{6} - 2\ln 6\right) - \left(\frac{3}{2} - 2\ln 2\right)\right]$$
$$= 36\pi\left(\frac{13}{3} + 2\ln\frac{1}{3}\right)$$
$$= 12\pi(13 - 6\ln 3)$$
$$\approx 241.59$$



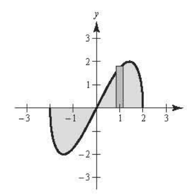
23. $R(x) = \dfrac{1}{\sqrt{x+1}},\ r(x) = 0$

$$V = \pi\int_0^3 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx$$
$$= \pi\int_0^3 \frac{1}{x+1}\, dx$$
$$= \Big[\pi\ln|x+1|\Big]_0^3$$
$$= \pi\ln 4 \approx 4.355$$

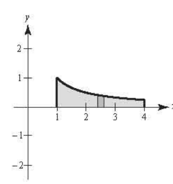


24. $R(x) = x\sqrt{4 - x^2},\ r(x) = 0$

$$V = 2\pi\int_0^2 \left[x\sqrt{4 - x^2}\right]^2 dx$$
$$= 2\pi\int_0^2 (4x^2 - x^4)\, dx$$
$$= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= \frac{128\pi}{15}$$



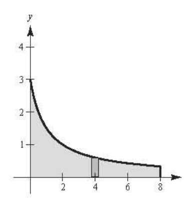
25. $R(x) = \dfrac{1}{x},\ r(x) = 0$



$$V = \pi\int_1^4 \left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left[-\frac{1}{x}\right]_1^4$$
$$= \frac{3\pi}{4}$$

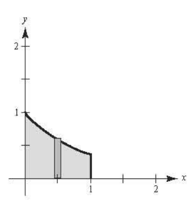
26. $R(x) = \dfrac{3}{x+1},\ r(x) = 0$

$$V = \pi\int_0^8 \left(\frac{3}{x+1}\right)^2 dx$$
$$= 9\pi\int_0^8 (x+1)^{-2}\, dx$$
$$= 9\pi\left[-\frac{1}{x+1}\right]_0^8 = 8\pi$$



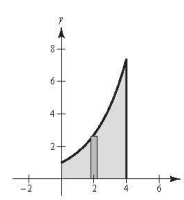
27. $R(x) = e^{-x},\ r(x) = 0$

$$V = \pi\int_0^1 (e^{-x})^2\, dx$$
$$= \pi\int_0^1 e^{-2x}\, dx$$
$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$
$$= \frac{\pi}{2}(1 - e^{-2}) \approx 1.358$$



28. $R(x) = e^{x/2},\ r(x) = 0$

$$V = \pi\int_0^4 (e^{x/2})^2\, dx$$
$$= \pi\int_0^4 e^x\, dx$$
$$= \Big[\pi e^x\Big]_0^4$$
$$= \pi(e^4 - 1) \approx 168.38$$



29. $x^2 + 1 = -x^2 + 2x + 5$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

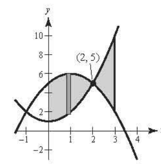
The curves intersect at $(-1, 2)$ and $(2, 5)$.

$$V = \pi\int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2)]\,dx + \pi\int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2]\,dx$$

$$= \pi\int_0^2 (-4x^3 - 8x^2 + 20x + 24)\,dx + \pi\int_2^3 (4x^3 + 8x^2 - 20x - 24)\,dx$$

$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$

$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

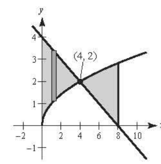


30. $$V = \pi\int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi\int_4^8 \left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx$$

$$= \pi\int_0^4 \left(\frac{x^2}{4} - 5x + 16\right) dx + \pi\int_4^8 \left(-\frac{x^2}{4} + 5x - 16\right) dx$$

$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8$$

$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$



31. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

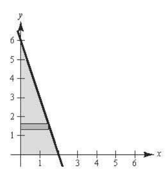
$$V = \pi\int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy$$

$$= \frac{\pi}{9}\int_0^6 [36 - 12y + y^2]\,dy$$

$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$

$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$

$$= 8\pi = \frac{1}{3}\pi r^2 h, \quad \text{Volume of cone}$$



32. $y = 9 - x^2,\ y = 0,\ x = 2,\ x = 3$

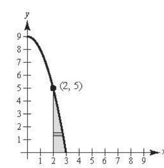
$x = \sqrt{9 - y}$

$$V = \pi\int_0^5 \left(\sqrt{9 - y}\right)^2 - 2]^2\,dy$$

$$= \pi\int_0^5 (5 - y)\,dy$$

$$= \pi\left[5y - \frac{y^2}{2}\right]_0^5$$

$$= \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2}$$



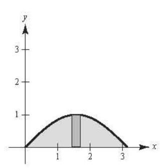
33. $$V = \pi\int_0^\pi (\sin x)^2\,dx$$

$$= \pi\int_0^\pi \frac{1 - \cos 2x}{2}\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$

$$= \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

Numerical approximation: 4.9348



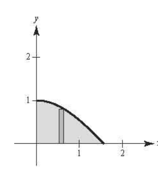
34. $$V = \pi\int_0^{\pi/2} [\cos x]^2\,dx$$

$$= \pi\int_0^{\pi/2} \frac{1 + \cos 2x}{2}\,dx$$

$$= \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$

Numerical approximation: 2.4674



35. $V = \pi \int_1^2 (e^{x-1})^2 \, dx$

$= \pi \int_1^2 e^{2x-2} \, dx$

$= \frac{\pi}{2} e^{2x-2} \Big]_1^2$

$= \frac{\pi}{2}(e^2 - 1)$

Numerical approximation: 10.0359

36. $V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2 \, dx$

$= \pi \int_{-1}^2 [e^x + e^{-x} + 2] \, dx$

$= \pi \Big[e^x - e^{-x} + 2x \Big]_{-1}^2$

$= \pi[(e^2 - e^{-2} + 4) - (e^{-1} - e - 2)]$

$= \pi[e^2 + e + 6 - e^{-2} - e^{-1}]$

Numerical approximation: 49.0218

37. $V = \pi \int_0^2 [e^{-x^2}]^2 \, dx \approx 1.9686$

38. $V = \pi \int_1^3 [\ln x]^2 \, dx \approx 3.2332$

39. $V = \pi \int_0^5 [2 \arctan(0.2x)]^2 \, dx$

≈ 15.4115

40. $x^2 = \sqrt{2x}$

$x^4 = 2x$

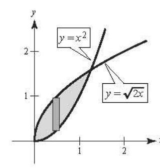
$x^3 = 2$

$x = 2^{1/3} \approx 1.2599$

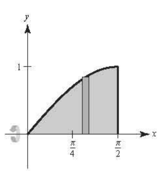
$V = \pi \int_0^{2^{1/3}} \left[\left(\sqrt{2x}\right)^2 - (x^2)^2 \right] dx$

$= \pi \int_0^{2^{1/3}} (2x - x^4) \, dx$

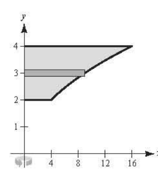
$= \frac{3 \cdot 2^{2/3} \pi}{5} \approx 2.9922$



41. $\pi \int_0^{\pi/2} \sin^2 x \, dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi/2$ about the x-axis.

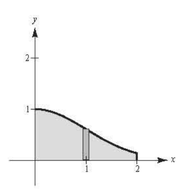


42. $\pi \int_2^4 y^4 \, dy$ represents the volume of the solid generated by revolving the region bounded by $x = y^2$, $x = 0$, $y = 2$, $y = 4$ about the y-axis.



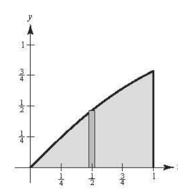
43. $A \approx 3$

Matches (a)

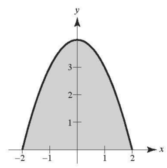


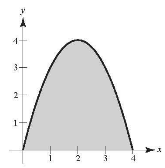
44. $A \approx \frac{3}{4}$

Matches (b)



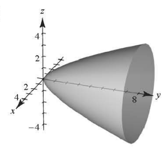
45.



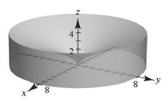


The volumes are the same because the solid has been translated horizontally. $(4x - x^2 = 4 - (x - 2)^2)$

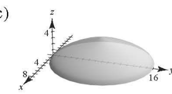
46. (a)



(b)



(c)



$a < c < b$

47. $R(x) = \frac{1}{2}x,\ r(x) = 0$

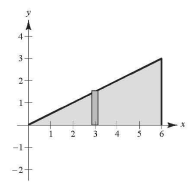
$$V = \pi\int_0^6 \frac{1}{4}x^2\,dx$$

$$= \left[\frac{\pi}{12}x^3\right]_0^6 = 18\pi$$

Note: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$

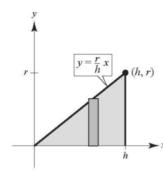


48. $R(x) = \frac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx$$

$$= \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h$$

$$= \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$



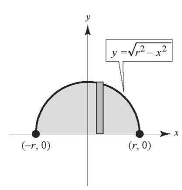
49. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$V = \pi\int_{-r}^{r} (r^2 - x^2)\,dx$$

$$= 2\pi\int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi\left[r^2x - \frac{1}{3}x^3\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$$



50. $x = \sqrt{r^2 - y^2},\ R(y) = \sqrt{r^2 - y^2},\ r(y) = 0$

$$V = \pi\int_h^r \left(\sqrt{r^2 - y^2}\right)^2 dy$$

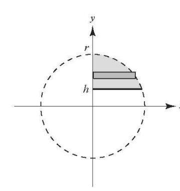
$$= \pi\int_h^r (r^2 - y^2)\,dy$$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_h^r$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2h - \frac{h^3}{3}\right)\right]$$

$$= \pi\left(\frac{2r^3}{3} - r^2h + \frac{h^3}{3}\right)$$

$$= \frac{\pi}{3}(2r^3 - 3r^2h + h^3)$$



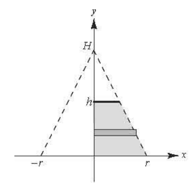
51. $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right)$, $R(y) = r\left(1 - \frac{y}{H}\right)$, $r(y) = 0$

$$V = \pi\int_0^h \left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right) dy$$

$$= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h$$

$$= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right)$$

$$= \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$$



52. (a) $V = \pi\int_0^4 (\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = \left[\frac{\pi x^2}{2}\right]_0^4 = 8\pi$

Let $0 < c < 4$ and set

$$\pi\int_0^c x\,dx = \left[\frac{\pi x^2}{2}\right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$c^2 = 8$

$c = \sqrt{8} = 2\sqrt{2}$

Thus, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

(b) Set $\pi\int_0^c x\,dx = \frac{8\pi}{3}$ (one third of the volume). Then

$\frac{\pi c^2}{2} = \frac{8\pi}{3}$, $c^2 = \frac{16}{3}$, $c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$.

To find the other value, set

$\pi\int_0^d x\,dx = \frac{16\pi}{3}$ (two thirds of the volume).

Then $\frac{\pi d^2}{2} = \frac{16\pi}{3}$, $d^2 = \frac{32}{3}$, $d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}$.

The x-values that divide the solid into three parts of equal volume are $x = (4\sqrt{3})/3$ and $x = (4\sqrt{6})/3$.

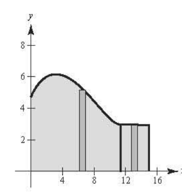
53. $V = \pi\int_0^2 \left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\,dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}$

54. $y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$

$$V = \pi\int_0^{11.5} \left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\,dx$$

$$= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2 x\right]_{11.5}^{15}$$

≈ 1031.9016 cubic centimeters



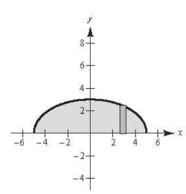
55. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2}$, $r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^5 (25 - x^2)\,dx$$

$$= \frac{18\pi}{25}\int_0^5 (25 - x^2)\,dx$$

$$= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5$$

$$= 60\pi$$

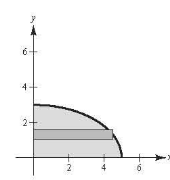


(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2}$, $r(y) = 0$, $x \ge 0$

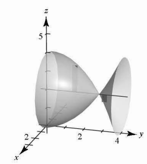
$$V = \frac{25\pi}{9}\int_0^3 (9 - y^2)\,dy$$

$$= \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3$$

$$= 50\pi$$



56. (a) First find where $y = b$ intersects the parabola:



$$b = 4 - \frac{x^2}{4}$$

$$x^2 = 16 - 4b = 4(4 - b)$$

$$x = 2\sqrt{4 - b}$$

$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^{4} \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$

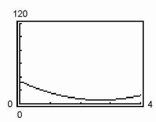
$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$

$$= \pi\int_0^4 \left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right] dx$$

$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$

$$= \pi\left[\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right] = \pi\left[4b^2 - \frac{64}{3}b + \frac{512}{15}\right]$$

(b) Graph of $V(b) = \pi\left[4b^2 - \frac{64}{3}b + \frac{512}{15}\right]$

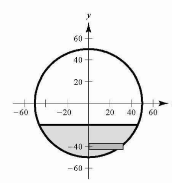


Minimum volume is 17.87 for $b = 2.67$.

(c) $V'(b) = \pi\left[8b - \frac{64}{3}\right] = 0 \implies b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \implies b = \frac{8}{3}$ is a relative minimum.

57. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500{,}000\pi}{3}$ ft^3



Volume of water in the tank:

$$\pi\int_{-50}^{y_0} \left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0} (2500 - y^2)\, dy$$

$$= \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0}$$

$$= \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

When the tank is one-fourth of its capacity:

$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$

$$y_0^3 - 7500y_0 - 125{,}000 = 0$$

$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

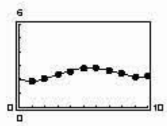
58. (a) $V = \int_0^{10} \pi[f(x)]^2\,dx$

Simpson's Rule: $b - a = 10 - 0 = 10,\ n = 10$

$$V \approx \frac{\pi}{3}[(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2]$$

$$\approx \frac{\pi}{3}[178.405] \approx 186.83\ \text{cm}^3$$

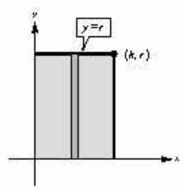
(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$



(c) $V \approx \int_0^{10} \pi f(x)^2\,dx \approx 186.35\ \text{cm}^3$

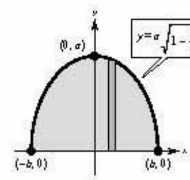
59. (a) $\pi\int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



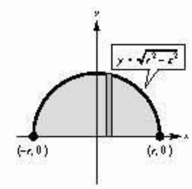
(b) $\pi\int_{-b}^{b}\left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



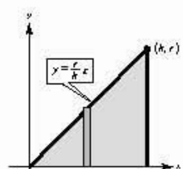
(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



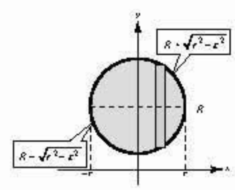
(d) $\pi\int_0^h \left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.



(e) $\pi\int_{-r}^{r}\left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.

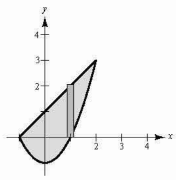


60. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2.$$

Thus, the volumes are the same.

61.



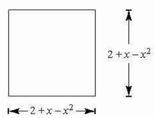
Base of cross section $= (x + 1) - (x^2 - 1) = 2 + x - x^2$

(a) $A(x) = b^2 = (2 + x - x^2)^2$

$= 4 + 4x - 3x^2 - 2x^3 + x^4$

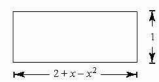
$V = \int_{-1}^{2} (4 + 4x - 3x^2 - 2x^3 + x^4)\,dx$

$= \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$

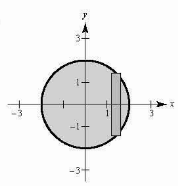


(b) $A(x) = bh = (2 + x - x^2)1$

$V = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$



62.

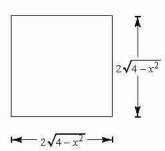


Base of cross section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$V = \int_{-2}^{2} 4(4 - x^2)\,dx$

$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{128}{3}$



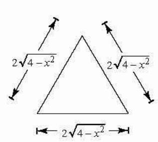
(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right)$

$= \sqrt{3}(4 - x^2)$

$V = \sqrt{3}\int_{-2}^{2} (4 - x^2)\,dx$

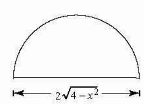
$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= \frac{32\sqrt{3}}{3}$



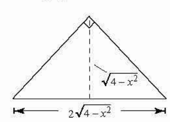
(c) $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}(4 - x^2)$

$V = \frac{\pi}{2}\int_{-2}^{2} (4 - x^2)\,dx = \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16\pi}{3}$

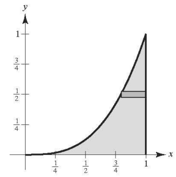


(d) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$

$V = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$



63.



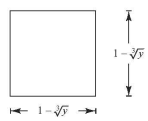
Base of cross section $= 1 - \sqrt[3]{y}$

(a) $A(y) = b^2 = \left(1 - \sqrt[3]{y}\right)^2$

$$V = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy$$

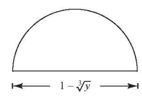
$$= \int_0^1 \left(1 - 2y^{1/3} + y^{2/3}\right) dy$$

$$= \left[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3}\right]_0^1 = \frac{1}{10}$$



(b) $A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1 - \sqrt[3]{y}}{2}\right)^2 = \frac{1}{8}\pi\left(1 - \sqrt[3]{y}\right)^2$

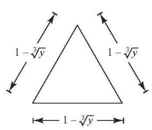
$$V = \frac{1}{8}\pi\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{8}\left(\frac{1}{10}\right) = \frac{\pi}{80}$$



(c) $A(y) = \frac{1}{2}bh = \frac{1}{2}\left(1 - \sqrt[3]{y}\right)\left(\frac{\sqrt{3}}{2}\right)\left(1 - \sqrt[3]{y}\right)$

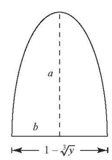
$$= \frac{\sqrt{3}}{4}\left(1 - \sqrt[3]{y}\right)^2$$

$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\sqrt{3}}{4}\left(\frac{1}{10}\right) = \frac{\sqrt{3}}{40}$$



(d) $A(y) = \frac{1}{2}\pi ab = \frac{\pi}{2}(2)\left(1 - \sqrt[3]{y}\right)\frac{1 - \sqrt[3]{y}}{2} = \frac{\pi}{2}\left(1 - \sqrt[3]{y}\right)^2$

$$V = \frac{\pi}{2}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{2}\left(\frac{1}{10}\right) = \frac{\pi}{20}$$



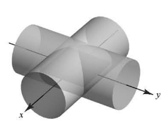
64. The cross sections are squares. By symmetry, we can set up an integral for an eighth of the volume and multiply by 8.

$A(y) = b^2 = \left(\sqrt{r^2 - y^2}\right)^2$

$$V = 8\int_0^r \left(r^2 - y^2\right) dy$$

$$= 8\left[r^2 y - \frac{1}{3}y^3\right]_0^r$$

$$= \frac{16}{3}r^3$$



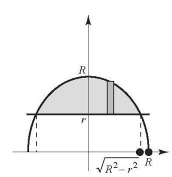
65. $V = \pi\int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx$

$$= 2\pi\int_0^{\sqrt{R^2 - r^2}} \left(R^2 - r^2 - x^2\right) dx$$

$$= 2\pi\left[\left(R^2 - r^2\right)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2 - r^2}}$$

$$= 2\pi\left[\left(R^2 - r^2\right)^{3/2} - \frac{\left(R^2 - r^2\right)^{3/2}}{3}\right]$$

$$= \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$$



66. $\frac{4}{3}\pi(25 - r^2)^{3/2} = \frac{1}{2}\left(\frac{4}{3}\right)\pi(125)$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2}\right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. $V = \pi\int_0^1 y^2\,dy = \pi\frac{y^3}{3}\Big]_0^1 = \frac{\pi}{3}$

68. $V = \pi\int_0^1 [1^2 - (1 - y)^2]\,dy$

$$= \pi\int_0^1 [2y - y^2]\,dy$$

$$= \pi\left[y^2 - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[1 - \frac{1}{3}\right] = \frac{2}{3}\pi$$

69. $V = \pi\int_0^1 (x^2 - x^4)\,dx$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{2\pi}{15}$$

70. $V = \pi\int_0^1 [(1 - x^2)^2 - (1 - x)^2]\,dx$

$$= \pi\int_0^1 [1 - 2x^2 + x^4 - 1 + 2x - x^2]\,dx$$

$$= \pi\int_0^1 [2x - 3x^2 + x^4]\,dx$$

$$= \pi\left[x^2 - x^3 + \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\frac{1}{5}\right] = \frac{\pi}{5}$$

71. $V = \pi\int_0^1 (1 - y)\,dy$

$$= \pi\left[y - \frac{y^2}{2}\right]_0^1$$

$$= \pi\left[1 - \frac{1}{2}\right]$$

$$= \frac{\pi}{2}$$

72. $V = \pi\int_0^1 \left(1 - \sqrt{y}\right)^2\,dy$

$$= \pi\int_0^1 \left(1 - 2\sqrt{y} + y\right)\,dy$$

$$= \pi\left[y - \frac{4}{3}y^{3/2} + \frac{y^2}{2}\right]_0^1$$

$$= \pi\left[1 - \frac{4}{3} + \frac{1}{2}\right]$$

$$= \frac{\pi}{6}$$

73. $V = \pi\int_0^1 (y - y^2)\,dy$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{\pi}{6}$$

74. $V = \pi \int_0^1 \left[(1-y)^2 - \left(1-\sqrt{y}\right)^2\right] dy$

$= \pi \int_0^1 \left[1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right] dy$

$= \pi \int_0^1 \left[2\sqrt{y} - 3y + y^2\right] dy$

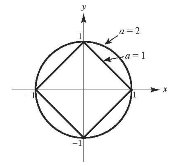
$= \pi \left[\frac{4}{3}y^{3/2} - \frac{3y^2}{2} + \frac{y^3}{3}\right]_0^1$

$= \pi \left[\frac{4}{3} - \frac{3}{2} + \frac{1}{3}\right]$

$= \frac{\pi}{6}$

75. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.

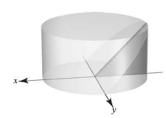


(b) $|y| = (1 - |x|^a)^{1/a}$

$A = 2\int_{-1}^{1} (1 - |x|^a)^{1/a}\,dx = 4\int_0^1 (1 - x^a)^{1/a}\,dx$

To approximate the volume of the solid, form n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.

76. (a) Since the cross sections are isosceles right triangles:



$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2 - y^2}\right)\left(\sqrt{r^2 - y^2}\right) = \frac{1}{2}(r^2 - y^2)$

$V = \frac{1}{2}\int_{-r}^{r} (r^2 - y^2)\,dy = \int_0^r (r^2 - y^2)\,dy = \left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3$

(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}\left(\sqrt{r^2 - y^2}\tan\theta\right) = \frac{\tan\theta}{2}(r^2 - y^2)$

$V = \frac{\tan\theta}{2}\int_{-r}^{r} (r^2 - y^2)\,dy = \tan\theta \int_0^r (r^2 - y^2)\,dy = \tan\theta\left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3 \tan\theta$

As $\theta \to 90^\circ$, $V \to \infty$.

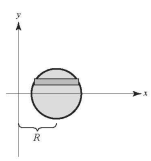
77. (a) $(x - R)^2 + y^2 = r^2$

$x = R \pm \sqrt{r^2 - y^2}$

$V = 2\pi \int_0^r \left(\left[R + \sqrt{r^2 - y^2}\right]^2 - \left[R - \sqrt{r^2 - y^2}\right]^2\right) dy$

$= 2\pi \int_0^r 4R\sqrt{r^2 - y^2}\,dy$

$= 8\pi R \int_0^r \sqrt{r^2 - y^2}\,dy$



(b) $\int_0^r \sqrt{r^2 - y^2}\,dy$ is one-quarter of the area of a circle of radius r, $\frac{1}{4}\pi r^2$.

$V = 8\pi R\left(\frac{1}{4}\pi r^2\right) = 2\pi^2 r^2 R$

Section 7.3 Volume: The Shell Method

1. $p(x) = x,\ h(x) = x$

$$V = 2\pi\int_0^2 x(x)\,dx$$
$$= \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

2. $p(x) = x,\ h(x) = 1 - x$

$$V = 2\pi\int_0^1 x(1-x)\,dx$$
$$= 2\pi\int_0^1 (x - x^2)\,dx$$
$$= 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

3. $p(x) = x,\ h(x) = \sqrt{x}$

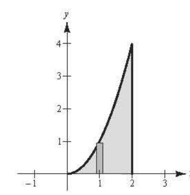
$$V = 2\pi\int_0^4 x\sqrt{x}\,dx$$
$$= 2\pi\int_0^4 x^{3/2}\,dx$$
$$= \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

4. $p(x) = x,\ h(x) = 8 - (x^2 + 4) = 4 - x^2$

$$V = 2\pi\int_0^2 x(4 - x^2)\,dx$$
$$= 2\pi\int_0^2 (4x - x^3)\,dx$$
$$= 2\pi\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8\pi$$

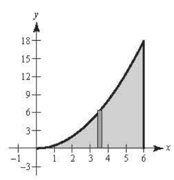
5. $p(x) = x,\ h(x) = x^2$

$$V = 2\pi\int_0^2 x^3\,dx$$
$$= \left[\frac{\pi}{2}x^4\right]_0^2 = 8\pi$$



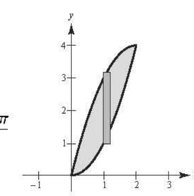
6. $p(x) = x,\ h(x) = \frac{1}{2}x^2$

$$V = 2\pi\int_0^6 \frac{1}{2}x^3\,dx$$
$$= \left[\pi\frac{x^4}{4}\right]_0^6 = 324\pi$$



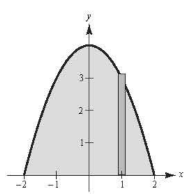
7. $p(x) = x,\ h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 x(4x - 2x^2)\,dx$$
$$= 4\pi\int_0^2 (2x^2 - x^3)\,dx$$
$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$



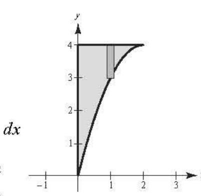
8. $p(x) = x,\ h(x) = 4 - x^2$

$$V = 2\pi\int_0^2 (4x - x^3)\,dx$$
$$= 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 8\pi$$



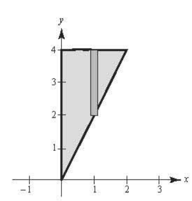
9. $p(x) = x$

$$h(x) = 4 - (4x - x^2)$$
$$= x^2 - 4x + 4$$
$$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2$$
$$= \frac{8\pi}{3}$$



10. $p(x) = x,\ h(x) = 4 - 2x$

$$V = 2\pi\int_0^2 x(4 - 2x)\,dx$$
$$= 2\pi\int_0^2 (4x - 2x^2)\,dx$$
$$= 2\pi\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{16\pi}{3}$$



Section 7.3 Volume: The Shell Method

1. $p(x) = x,\ h(x) = x$

$$V = 2\pi\int_0^2 x(x)\,dx$$
$$= \left[\frac{2\pi x^3}{3}\right]_0^2 = \frac{16\pi}{3}$$

2. $p(x) = x,\ h(x) = 1 - x$

$$V = 2\pi\int_0^1 x(1 - x)\,dx$$
$$= 2\pi\int_0^1 (x - x^2)\,dx$$
$$= 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

3. $p(x) = x,\ h(x) = \sqrt{x}$

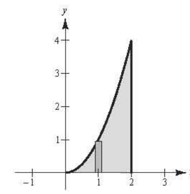
$$V = 2\pi\int_0^4 x\sqrt{x}\,dx$$
$$= 2\pi\int_0^4 x^{3/2}\,dx$$
$$= \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

4. $p(x) = x,\ h(x) = 8 - (x^2 + 4) = 4 - x^2$

$$V = 2\pi\int_0^2 x(4 - x^2)\,dx$$
$$= 2\pi\int_0^2 (4x - x^3)\,dx$$
$$= 2\pi\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8\pi$$

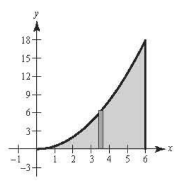
5. $p(x) = x,\ h(x) = x^2$

$$V = 2\pi\int_0^2 x^3\,dx$$
$$= \left[\frac{\pi}{2}x^4\right]_0^2 = 8\pi$$



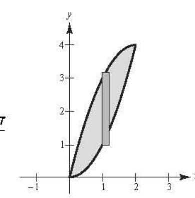
6. $p(x) = x,\ h(x) = \frac{1}{2}x^2$

$$V = 2\pi\int_0^6 \frac{1}{2}x^3\,dx$$
$$= \left[\pi\frac{x^4}{4}\right]_0^6 = 324\pi$$



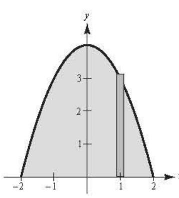
7. $p(x) = x,\ h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 x(4x - 2x^2)\,dx$$
$$= 4\pi\int_0^2 (2x^2 - x^3)\,dx$$
$$= 4\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16\pi}{3}$$



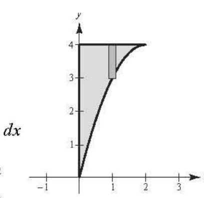
8. $p(x) = x,\ h(x) = 4 - x^2$

$$V = 2\pi\int_0^2 (4x - x^3)\,dx$$
$$= 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 8\pi$$



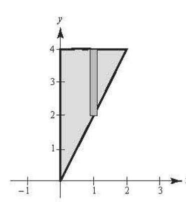
9. $p(x) = x$

$$h(x) = 4 - (4x - x^2)$$
$$= x^2 - 4x + 4$$
$$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2$$
$$= \frac{8\pi}{3}$$



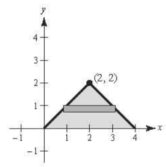
10. $p(x) = x,\ h(x) = 4 - 2x$

$$V = 2\pi\int_0^2 x(4 - 2x)\,dx$$
$$= 2\pi\int_0^2 (4x - 2x^2)\,dx$$
$$= 2\pi\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{16\pi}{3}$$



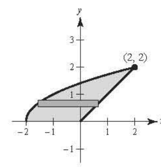
19. $p(y) = y,\ h(y) = (4 - y) - (y) = 4 - 2y$

$$V = 2\pi\int_0^2 y(4 - 2y)\,dy$$
$$= 2\pi\int_0^2 (4y - 2y^2)\,dy$$
$$= 2\pi\left[2y^2 - \frac{2}{3}y^3\right]_0^2$$
$$= 2\pi\left[8 - \frac{16}{3}\right] = \frac{16\pi}{3}$$



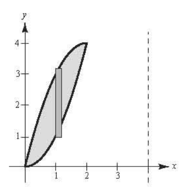
20. $p(y) = y,\ h(y) = y - (y^2 - 2) = 2 + y - y^2$

$$V = 2\pi\int_0^2 y(2 + y - y^2)\,dy$$
$$= 2\pi\int_0^2 (2y + y^2 - y^3)\,dy$$
$$= 2\pi\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$
$$= 2\pi\left[4 + \frac{8}{3} - 4\right] = \frac{16\pi}{3}$$



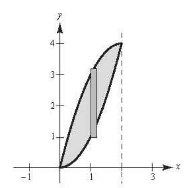
21. $p(x) = 4 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 (4 - x)(4x - 2x^2)\,dx$$
$$= 2\pi(2)\int_0^2 (x^3 - 6x^2 + 8x)\,dx$$
$$= 4\pi\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 16\pi$$



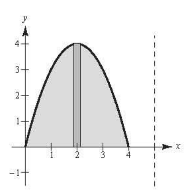
22. $p(x) = 2 - x,\ h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi\int_0^2 (2 - x)(4x - 2x^2)\,dx$$
$$= 2\pi\int_0^2 (8x - 8x^2 + 2x^3)\,dx$$
$$= 2\pi\left[4x^2 - \frac{8}{3}x^3 + \frac{1}{2}x^4\right]_0^2 = \frac{16\pi}{3}$$



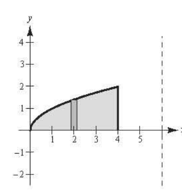
23. $p(x) = 5 - x,\ h(x) = 4x - x^2$

$$V = 2\pi\int_0^4 (5 - x)(4x - x^2)\,dx$$
$$= 2\pi\int_0^4 (x^3 - 9x^2 + 20x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 64\pi$$



24. $p(x) = 6 - x,\ h(x) = \sqrt{x}$

$$V = 2\pi\int_0^4 (6 - x)\sqrt{x}\,dx$$
$$= 2\pi\int_0^4 (6x^{1/2} - x^{3/2})\,dx$$
$$= 2\pi\left[4x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 = \frac{192\pi}{5}$$



25. The shell method would be easier: $V = 2\pi\int_0^4 \left[4 - (y - 2)^2\right]y\,dy$ shells

Using the disk method: $V = \pi\int_0^4 \left[\left(2 + \sqrt{4 - x}\right)^2 - \left(2 - \sqrt{4 - x}\right)^2\right]dx$ $\left[\textbf{Note: } V = \frac{128\pi}{3}\right]$

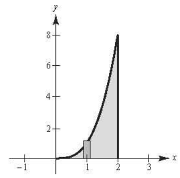
26. The shell method is easier: $V = 2\pi\int_0^{\ln 4} x(4 - e^x)\, dx$

Using the disk method, $x = \ln(4 - y)$ and $V = \pi\int_0^3 (\ln(4 - y))^2\, dy$. [**Note:** $V = \pi[8(\ln 2)^2 - 8\ln 2 + 3]$]

27. (a) **Disk**

$R(x) = x^3,\ r(x) = 0$

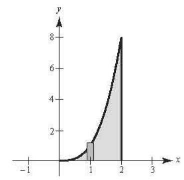
$V = \pi\int_0^2 x^6\, dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



(b) **Shell**

$p(x) = x,\ h(x) = x^3$

$V = 2\pi\int_0^2 x^4\, dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$



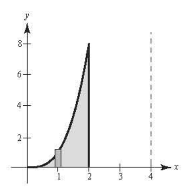
(c) **Shell**

$p(x) = 4 - x,\ h(x) = x^3$

$V = 2\pi\int_0^2 (4 - x)x^3\, dx$

$= 2\pi\int_0^2 (4x^3 - x^4)\, dx$

$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$



28. (a) **Disk**

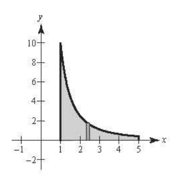
$R(x) = \frac{10}{x^2},\ r(x) = 0$

$V = \pi\int_1^5 \left(\frac{10}{x^2}\right)^2 dx$

$= 100\pi\int_1^5 x^{-4}\, dx$

$= 100\pi\left[\frac{x^{-3}}{-3}\right]_1^5$

$= -\frac{100\pi}{3}\left[\frac{1}{125} - 1\right] = \frac{496}{15}\pi$



(b) **Shell**

$R(x) = x,\ r(x) = 0$

$V = 2\pi\int_1^5 x\left(\frac{10}{x^2}\right) dx$

$= 20\pi\int_1^5 \frac{1}{x}\, dx$

$= 20\pi\Big[\ln|x|\Big]_1^5 = 20\pi\ln 5$

(c) **Disk**

$R(x) = 10,\ r(x) = 10 - \frac{10}{x^2}$

$V = \pi\int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2}\right)^2\right] dx$

$= \pi\left[\frac{100}{3x^3} - \frac{200}{x}\right]_1^5 = \frac{1904}{15}\pi$

29. (a) **Shell**

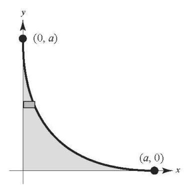
$p(y) = y,\ h(y) = (a^{1/2} - y^{1/2})^2$

$$V = 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y)\,dy$$

$$= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2)\,dy$$

$$= 2\pi\left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi\left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right] = \frac{\pi a^3}{15}$$



(b) Same as part (a) by symmetry

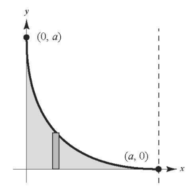
(c) **Shell**

$p(x) = a - x,\ h(x) = (a^{1/2} - x^{1/2})^2$

$$V = 2\pi\int_0^a (a - x)(a^{1/2} - x^{1/2})^2\,dx$$

$$= 2\pi\int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2)\,dx$$

$$= 2\pi\left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a = \frac{4\pi a^3}{15}$$



30. (a) **Disk**

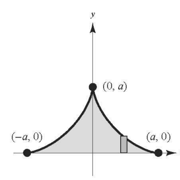
$R(x) = (a^{2/3} - x^{2/3})^{3/2},\ \ r(x) = 0$

$$V = \pi\int_{-a}^a (a^{2/3} - x^{2/3})^3\,dx$$

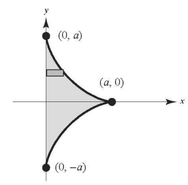
$$= 2\pi\int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)\,dx$$

$$= 2\pi\left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3\right]_0^a$$

$$= 2\pi\left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3\right) = \frac{32\pi a^3}{105}$$



(b) Same as part (a) by symmetry

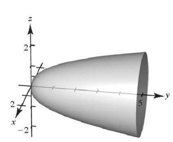


31. Answers will vary.

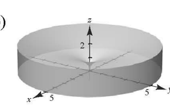
(a) The rectangles would be vertical.

(b) The rectangles would be horizontal.

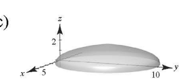
32. (a)



(b)



(c)



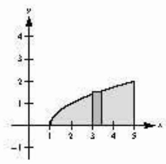
$a < c < b$

33. $\pi\int_1^5 (x-1)\,dx = \pi\int_1^5 \left(\sqrt{x-1}\right)^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y=\sqrt{x-1}$, $y=0$, and $x=5$ about the x-axis by using the disk method.

$$2\pi\int_0^2 y[5-(y^2+1)]\,dy$$

represents this same volume by using the shell method.



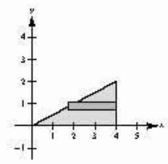
Disk method

34. $2\pi\int_0^4 x\left(\frac{x}{2}\right)dx$

represents the volume of the solid generated by revolving the region bounded by $y=x/2$, $y=0$, and $x=4$ about the y-axis by using the shell method.

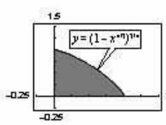
$$\pi\int_0^2 [16-(2y)^2]\,dy = \pi\int_0^2 [(4)^2-(2y)^2]\,dy$$

represents this same volume by using the disk method.



Disk method

35. (a)

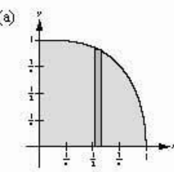


(b) $x^{4/3}+y^{4/3}=1$, $x=0$, $y=0$

$y=(1-x^{4/3})^{3/4}$

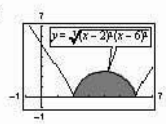
$V = 2\pi\int_0^1 x(1-x^{4/3})^{3/4}\,dx \approx 1.5056$

36. (a)



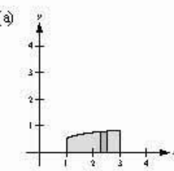
(b) $V = 2\pi\int_0^1 x\sqrt{1-x^3}\,dx \approx 2.3222$

37. (a)



(b) $V = 2\pi\int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2}\,dx \approx 187.249$

38. (a)

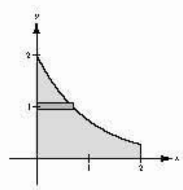


(b) $V = 2\pi\int_1^3 \frac{2x}{1+e^{1/x}}\,dx \approx 19.0162$

39. $y=2e^{-x}$, $y=0$, $x=0$, $x=2$

Volume ≈ 7.5

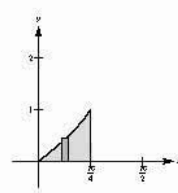
Matches (d)



40. $y=\tan x$, $y=0$, $x=0$, $x=\frac{\pi}{4}$

Volume ≈ 1

Matches (e)



41. $p(x) = x,\ h(x) = 2 - \frac{1}{2}x^2$

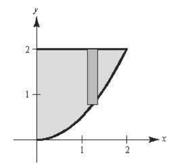
$$V = 2\pi\int_0^2 x\left(2 - \frac{1}{2}x^2\right)dx = 2\pi\int_0^2 \left(2x - \frac{1}{2}x^3\right)dx = 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \quad \text{(total volume)}$$

Now find x_0 such that:

$$\pi = 2\pi\int_0^{x_0}\left(2x - \frac{1}{2}x^3\right)dx$$

$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$



$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \qquad \text{(Quadratic Formula)}$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$, since the other root is too large.

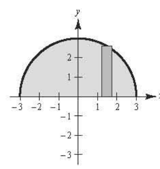
Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

42. Total volume of the hemisphere is $\frac{1}{2}\left(\frac{4}{3}\right)\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method, $p(x) = x$, $h(x) = \sqrt{9 - x^2}$. Find x_0 such that:

$$6\pi = 2\pi\int_0^{x_0} x\sqrt{9 - x^2}\,dx$$

$$6 = -\int_0^{x_0}(9 - x^2)^{1/2}(-2x)\,dx$$

$$= \left[-\frac{2}{3}(9 - x^2)^{3/2}\right]_0^{x_0} = 18 - \frac{2}{3}(9 - x_0^2)^{3/2}$$



$$(9 - x_0^2)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460$$

Diameter: $2\sqrt{9 - 18^{2/3}} \approx 2.920$

43. $V = 4\pi\int_{-1}^{1}(2 - x)\sqrt{1 - x^2}\,dx$

$$= 8\pi\int_{-1}^{1}\sqrt{1 - x^2}\,dx - 4\pi\int_{-1}^{1} x\sqrt{1 - x^2}\,dx$$

$$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi\int_{-1}^{1} x(1 - x^2)^{1/2}(-2)\,dx$$

$$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_{-1}^{1} = 4\pi^2$$

44. $V = 4\pi\int_{-r}^{r}(R - x)\sqrt{r^2 - x^2}\,dx$

$$= 4\pi R\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx - 4\pi\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx$$

$$= 4\pi R\left(\frac{\pi r^2}{2}\right) + \left[2\pi\left(\frac{2}{3}\right)(r^2 - x^2)^{3/2}\right]_{-r}^{r}$$

$$= 2\pi^2 r^2 R$$

45. (a) $\frac{d}{dx}[\sin x - x\cos x + C] = \cos x + x\sin x - \cos x = x\sin x$

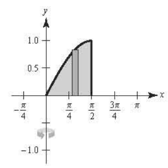
Hence, $\int x\sin x\,dx = \sin x - x\cos x + C.$

—CONTINUED—

45. —CONTINUED—

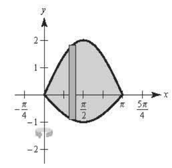
(b) (i) $p(x) = x, h(x) = \sin x$

$$V = 2\pi\int_0^{\pi/2} x \sin x \, dx$$
$$= 2\pi\Big[\sin x - x\cos x\Big]_0^{\pi/2}$$
$$= 2\pi[(1 - 0) - 0] = 2\pi$$



(ii) $p(x) = x, h(x) = 2\sin x - (-\sin x) = 3\sin x$

$$V = 2\pi\int_0^{\pi} x(3\sin x)\, dx$$
$$= 6\pi\int_0^{\pi} x\sin x\, dx$$
$$= 6\pi\Big[\sin x - x\cos x\Big]_0^{\pi}$$
$$= 6\pi[\pi] = 6\pi^2$$

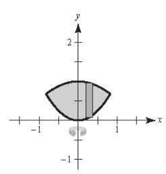


46. (a) $\frac{d}{dx}[\cos x + x\sin x + C] = -\sin x + \sin x + x\cos x = x\cos x$

Hence, $\int x\cos x\, dx = \cos x + x\sin x + C.$

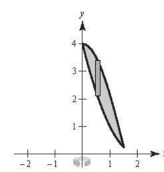
(b) (i) $x^2 = \cos x \implies x \approx \pm 0.8241$

$$V \approx 2(2\pi)\int_0^{0.8241} x[\cos x - x^2]\, dx$$
$$= 4\pi\left[\cos x + x\sin x - \frac{x^4}{4}\right]_0^{0.8241}$$
$$\approx 2.1205$$



(ii) $4\cos x = (x - 2)^2 \implies x = 0, 1.5110$

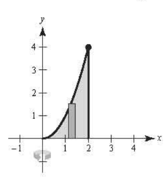
$$V \approx 2\pi\int_0^{1.511} x[4\cos x - (x - 2)^2]\, dx$$
$$= 2\pi\int_0^{1.511}\left[4\cos x + 4x\sin x - \frac{(x-2)^3}{3}\right]_0^{1.511}$$
$$= 6.2993$$



47. $2\pi\int_0^2 x^3\, dx = 2\pi\int_0^2 x(x^2)\, dx$

(a) Plane region bounded by $y = x^2, y = 0, x = 0, x = 2$

(b) Revolved about the y-axis

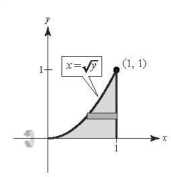


Other answers possible

48. $2\pi\int_0^1 (y - y^{3/2})\, dy = 2\pi\int_0^1 y(1 - \sqrt{y})\, dy$

(a) Plane region bounded by $x = \sqrt{y}, x = 1, y = 0$

(b) Revolved about the x-axis

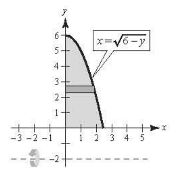


Other answers possible

49. $2\pi \int_0^6 (y+2)\sqrt{6-y}\,dy$

(a) Plane region bounded by $x = \sqrt{6-y}$, $x = 0$, $y = 0$

(b) Revolved around line $y = -2$

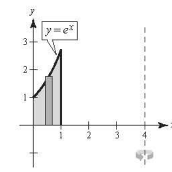


Other answers possible

50. $2\pi \int_0^1 (4-x)e^x\,dx$

(a) Plane region bounded by $y = e^x$, $y = 0$, $x = 0$, $x = 1$

(b) Revolved about the line $x = 4$



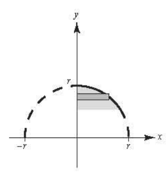
51. Disk Method

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy$$

$$= \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$$



52. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

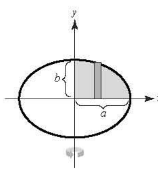
$$p(x) = x,\ h(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$V = 2(2\pi)\int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\,dx$$

$$= \frac{4\pi b}{a}\int_0^a \sqrt{a^2 - x^2}\,x\,dx$$

$$= \frac{4\pi b}{a}\left(\frac{-(a^2 - x^2)^{3/2}}{3}\right)\Bigg]_0^a$$

$$= \frac{4\pi b}{3a}a^3 = \frac{4}{3}\pi a^2 b$$



Note: If $a = b$, then volume is that of a sphere.

53. (a) Area region $= \int_0^b [ab^n - ax^n]\,dx$

$$= \left[ab^n x - a\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= ab^{n+1} - a\frac{b^{n+1}}{n+1}$$

$$= ab^{n+1}\left(1 - \frac{1}{n+1}\right) = ab^{n+1}\left(\frac{n}{n+1}\right)$$

$$R_1(n) = \frac{ab^{n+1}[n/(n+1)]}{(ab^n)b} = \frac{n}{n+1}$$

(b) $\lim_{n\to\infty} R_1(n) = \lim_{n\to\infty} \dfrac{n}{n+1} = 1$

$\lim_{n\to\infty} (ab^n)b = \infty$

(c) **Disk Method:**

$$V = 2\pi\int_0^b x(ab^n - ax^n)\,dx$$

$$= 2\pi a\int_0^b (xb^n - x^{n+1})\,dx$$

$$= 2\pi a\left[\frac{b^n}{2}x^2 - \frac{x^{n+2}}{n+2}\right]_0^b$$

$$= 2\pi a\left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\frac{n}{n+2}\right)$$

$$R_2(n) = \frac{\pi ab^{n+2}[n/(n+2)]}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2}\right)$$

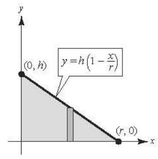
(d) $\lim_{n\to\infty} R_2(n) = \lim_{n\to\infty}\left(\dfrac{n}{n+2}\right) = 1$

$\lim_{n\to\infty} (\pi b^2)(ab^n) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = 1$.

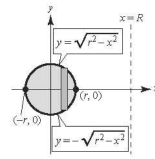
54. (a) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h.



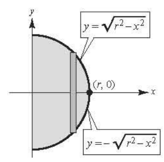
(b) $2\pi \int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.



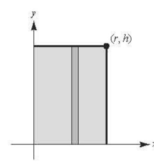
(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx$ (iii)

is the volume of a sphere with radius r.



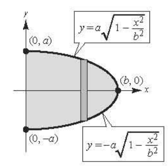
(d) $2\pi \int_0^r hx\, dx$ (i)

is the volume of a right circular cylinder with a radius of r and a height of h.



(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)}\, dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



55. (a) $V = 2\pi \int_0^4 xf(x)\, dx$

$= \frac{2\pi(40)}{3(4)}[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0]$

$= \frac{20\pi}{3}[5800] \approx 121{,}475$ cubic feet

(b) Top line: $y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \implies y = -\frac{1}{2}x + 50$

Bottom line: $y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \implies y = -2x + 80$

$V = 2\pi \int_0^{20} x\left(-\frac{1}{2}x + 50\right) dx + 2\pi \int_{20}^{40} x(-2x + 80)\, dx$

$= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x\right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x)\, dx$

$= 2\pi\left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40}$

$= 2\pi\left[\frac{26{,}000}{3}\right] + 2\pi\left[\frac{32{,}000}{3}\right]$

$\approx 121{,}475$ cubic feet

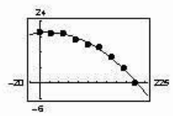
(Note that Simpson's Rule is exact for this problem.)

56. (a) $V = 2\pi\int_0^{200} xf(x)\,dx$

$\approx \frac{2\pi(200)}{3(8)}[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$

$\approx 1{,}366{,}593$ cubic feet

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi\int_0^{200} xd(x)\,dx \approx 2\pi(213{,}800)$

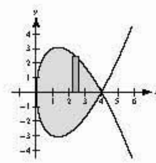
$= 1{,}343{,}345$ cubic feet

(d) Number gallons $\approx V(7.48) = 10{,}048{,}221$ gallons

57. $y^2 = x(4 - x)^2, \quad 0 \le x \le 4$

$y_1 = \sqrt{x(4-x)^2} = (4 - x)\sqrt{x}$

$y_2 = -\sqrt{x(4-x)^2} = -(4 - x)\sqrt{x}$



(a) $V = \pi\int_0^4 x(4 - x)^2\,dx$

$= \pi\int_0^4 (x^3 - 8x^2 + 16x)\,dx$

$= \pi\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^4 = \frac{64\pi}{3}$

(b) $V = 4\pi\int_0^4 x(4 - x)\sqrt{x}\,dx$

$= 4\pi\int_0^4 (4x^{3/2} - x^{5/2})\,dx$

$= 4\pi\left[\frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{2048\pi}{35}$

(c) $V = 4\pi\int_0^4 (4 - x)(4 - x)\sqrt{x}\,dx$

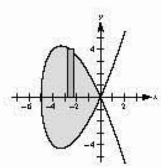
$= 4\pi\int_0^4 \left(16\sqrt{x} - 8x^{3/2} + x^{5/2}\right)dx$

$= 4\pi\left[\frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{8192\pi}{105}$

58. $y^2 = x^2(x + 5), \quad -5 \le x \le 0$

$y_1 = \sqrt{x^2(x+5)} = x\sqrt{x+5}$

$y_2 = -\sqrt{x^2(x+5)} = -x\sqrt{x+5}$



(a) $V = \pi\int_{-5}^0 x^2(x + 5)\,dx$

$= \pi\left[\frac{x^4}{4} + \frac{5x^3}{3}\right]_{-5}^0 = \frac{625\pi}{12}$

(b) $V = 4\pi\int_{-5}^0 x\left(x\sqrt{x+5}\right)dx$

Let $u = x + 5$, $du = dx$.

$V = 4\pi\int_0^5 (u - 5)^2\sqrt{u}\,du$

$= 4\pi\int_0^5 (u^{5/2} - 10u^{3/2} + 25u^{1/2})\,du$

$= 4\pi\left[\frac{2}{7}u^{7/2} - 4u^{5/2} + \frac{50}{3}u^{3/2}\right]_0^5 = \frac{1600\sqrt{5}\,\pi}{21}$

(c) $V = 4\pi\int_{-5}^0 (-5 - x)x\sqrt{x+5}\,dx$

Let $u = x + 5$, $du = dx$.

$V = 4\pi\int_0^5 (-u)(u - 5)\sqrt{u}\,du$

$= 4\pi\int_0^5 (-u^{5/2} + 5u^{3/2})\,du$

$= 4\pi\left[-\frac{2}{7}u^{7/2} + 2u^{5/2}\right]_0^5 = \frac{400\sqrt{5}\,\pi}{7}$

59. $V_1 = \pi\int_{1/4}^{c} \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_{1/4}^{c} = \pi\left[-\frac{1}{c} + 4\right] = \frac{4c-1}{c}\pi$

$V_2 = 2\pi\int_{1/4}^{c} x\left(\frac{1}{x}\right)dx = 2\pi x\Big]_{1/4}^{c} = 2\pi\left(c - \frac{1}{4}\right)$

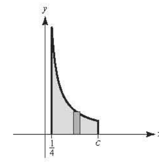
$V_1 = V_2 \implies \frac{4c-1}{c}\pi = 2\pi\left(c - \frac{1}{4}\right)$

$4c - 1 = 2c\left(c - \frac{1}{4}\right)$

$4c^2 - 9c + 2 = 0$

$(4c - 1)(c - 2) = 0$

$c = 2 \quad \left(c = \frac{1}{4} \text{ yields no volume.}\right)$



Section 7.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2}$

$= 13$

(b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2}\,dx$

$= \left[\frac{13}{5}x\right]_0^5 = 13$

2. $(1, 2), (7, 10)$

(a) $d = \sqrt{(7-1)^2 + (10-2)^2}$

$= 10$

(b) $y = \frac{4}{3}x + \frac{2}{3}$

$y' = \frac{4}{3}$

$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2}\,dx$

$= \left[\frac{5}{3}x\right]_1^7 = 10$

3. $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, \quad 0 \le x \le 1$

$s = \int_0^1 \sqrt{1 + x}\,dx$

$= \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^1$

$= \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$

4. $y = 2x^{3/2} + 3$

$y' = 3x^{1/2}, \quad 0 \le x \le 9$

$s = \int_0^9 \sqrt{1 + 9x}\,dx$

$= \left[\frac{2}{27}(1 + 9x)^{3/2}\right]_0^9$

$= \frac{2}{27}(82^{3/2} - 1) \approx 54.929$

5. $y = \frac{3}{2}x^{2/3}$

$y' = \frac{1}{x^{1/3}}, \quad 1 \le x \le 8$

$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx$

$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\,dx$

$= \frac{3}{2}\int_1^8 \sqrt{x^{2/3} + 1}\left(\frac{2}{3x^{1/3}}\right)dx$

$= \frac{3}{2}\left[\frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^8$

$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$

6. $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, \quad 1 \le x \le 2$

$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, \quad [1, 2]$

$s = \int_a^b \sqrt{1 + (y')^2}\,dx$

$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)dx$

$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2$

$= \frac{33}{16} \approx 2.063$

7. $y = \frac{x^5}{10} + \frac{1}{6x^3}$

$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$

$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, \quad 1 \le x \le 2$

$s = \int_a^b \sqrt{1 + (y')^2}\,dx$

$= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2}\,dx$

$= \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right) dx$

$= \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.2458$

8. $y = \frac{3}{2}x^{2/3} + 4$

$y' = x^{-1/3}, \quad 1 \le x \le 27$

$s = \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx$

$= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\,dx$

$= \frac{3}{2}\int_1^{27} \sqrt{x^{2/3} + 1}\left(\frac{2}{3x^{1/3}}\right) dx$

$= \left[\frac{3}{2} \cdot \frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^{27}$

$= 10^{3/2} - 2^{3/2} \approx 28.794$

9. $y = \ln(\sin x), \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$y' = \frac{1}{\sin x}\cos x = \cot x$

$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$

$s = \int_{\pi/4}^{3\pi/4} \csc x\,dx$

$= \left[\ln\left|\csc x - \cot x\right|\right]_{\pi/4}^{3\pi/4}$

$= \ln\left(\sqrt{2} + 1\right) - \ln\left(\sqrt{2} - 1\right) \approx 1.763$

10. $y = \ln(\cos x), \quad 0 \le x \le \frac{\pi}{3}$

$y' = \frac{-\sin x}{\cos x} = -\tan x$

$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$

$s = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx$

$= \int_0^{\pi/3} \sec x\,dx$

$= \ln\left|\sec x + \tan x\right|\Big]_0^{\pi/3}$

$= \ln\left(2 + \sqrt{3}\right) \approx 1.3170$

11. $y = \frac{1}{2}(e^x + e^{-x})$

$y' = \frac{1}{2}(e^x - e^{-x}), \quad [0, 2]$

$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, \quad [0, 2]$

$s = \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}\,dx$

$= \frac{1}{2}\int_0^2 (e^x + e^{-x})\,dx$

$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$

12. $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right) = \ln(e^x + 1) - \ln(e^x - 1)$

$\frac{dy}{dx} = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{-2e^x}{e^{2x} - 1} = \frac{2e^x}{1 - e^{2x}}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4e^{2x}}{1 - 2e^{2x} + e^{4x}}$

$= \frac{1 + 2e^{2x} + e^{4x}}{(1 - e^{2x})^2} = \left(\frac{1 + e^{2x}}{1 - e^{2x}}\right)^2$

$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{\ln 2}^{\ln 3} \frac{1 + e^{2x}}{e^{2x} - 1}\,dx$

$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int_{\ln 2}^{\ln 3} \coth x\,dx$

$= \ln\left(\sinh(x)\right)\Big]_{\ln 2}^{\ln 3} = \ln\left(\frac{4}{3}\right) - \ln\left(\frac{3}{4}\right)$

$= \ln\left(\frac{4/3}{3/4}\right) = \ln\frac{16}{9} - 2\ln\left(\frac{4}{3}\right) \approx 0.57536$

13. $x = \frac{1}{3}(y^2+2)^{3/2}, \quad 0 \le y \le 4$

$$\frac{dx}{dy} = y(y^2+2)^{1/2}$$

$$s = \int_0^4 \sqrt{1 + y^2(y^2+2)}\,dy$$

$$= \int_0^4 \sqrt{y^4 + 2y^2 + 1}\,dy$$

$$= \int_0^4 (y^2+1)\,dy$$

$$= \left[\frac{y^3}{3} + y\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

14. $x = \frac{1}{3}\sqrt{y}(y-3), \quad 1 \le y \le 4$

$$x = \frac{1}{3}(y^{3/2} - 3y^{1/2})$$

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y + \frac{1}{4}y^{-1} - \frac{1}{2}$$

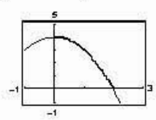
$$= \frac{1}{4}(y + 2 + y^{-1}) = \frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2$$

$$s = \int_1^4 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)dy$$

$$= \left[\frac{1}{2}\left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right)\right]_1^4$$

$$= \frac{1}{2}\left(\frac{16}{3} + 4\right) - \frac{1}{2}\left(\frac{2}{3} + 2\right) = \frac{10}{3}$$

15. (a) $y = 4 - x^2, \quad 0 \le x \le 2$



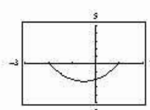
(b) $y' = -2x$

$$1 + (y')^2 = 1 + 4x^2$$

$$L = \int_0^2 \sqrt{1 + 4x^2}\,dx$$

(c) $L \approx 4.647$

16. (a) $y = x^2 + x - 2, \quad -2 \le x \le 1$



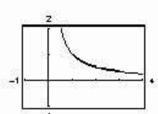
(b) $y' = 2x + 1$

$$1 + (y')^2 = 1 + 4x^2 + 4x + 1$$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\,dx$$

(c) $L \approx 5.653$

17. (a) $y = \frac{1}{x}, \quad 1 \le x \le 3$



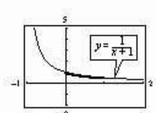
(b) $y' = -\frac{1}{x^2}$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\,dx$$

(c) $L \approx 2.147$

18. (a) $y = \frac{1}{1+x}, \quad 0 \le x \le 1$



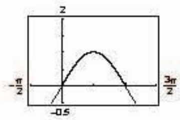
(b) $y' = -\frac{1}{(1+x)^2}$

$$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$$

$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}}\,dx$$

(c) $L \approx 1.132$

19. (a) $y = \sin x, \quad 0 \le x \le \pi$



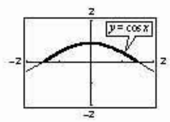
(b) $y' = \cos x$

$1 + (y')^2 = 1 + \cos^2 x$

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

20. (a) $y = \cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



(b) $y' = -\sin x$

$1 + (y')^2 = 1 + \sin^2 x$

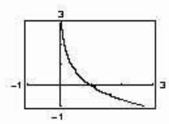
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

21. (a) $x = e^{-y}, \quad 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $y' = -\frac{1}{x}$

$1 + (y')^2 = 1 + \frac{1}{x^2}$

$$L = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y}, \quad 0 \le y \le 2$

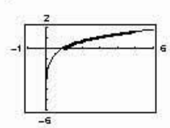
(b) $\frac{dx}{dy} = -e^{-y}$

$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$

$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

22. (a) $y = \ln x, \quad 1 \le x \le 5$



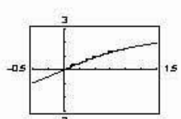
(b) $y' = \frac{1}{x}$

$1 + (y')^2 = 1 + \frac{1}{x^2}$

$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 4.367$

23. (a) $y = 2 \arctan x, \quad 0 \le x \le 1$



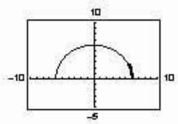
(b) $y' = \frac{2}{1 + x^2}$

$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\, dx$$

(c) $L \approx 1.871$

24. (a) $x = \sqrt{36 - y^2}, \quad 0 \le y \le 3$

$y = \sqrt{36 - x^2}, \quad 3\sqrt{3} \le x \le 6$



(b) $\dfrac{dx}{dy} = \dfrac{1}{2}(36 - y^2)^{-1/2}(-2y)$

$= \dfrac{-y}{\sqrt{36 - y^2}}$

$L = \displaystyle\int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\, dy$

$= \displaystyle\int_0^3 \frac{6}{\sqrt{36 - y^2}}\, dy$

(c) $L \approx 3.142 \quad (\pi!)$

Alternatively, you can convert to a function of x.

$y = \sqrt{36 - x^2}$

$y' = \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{36 - x^2}}$

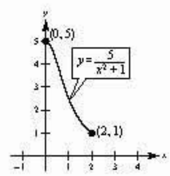
$L = \displaystyle\int_{3\sqrt{3}}^6 \sqrt{1 + \frac{x^2}{36 - x^2}}\, dx = \int_{3\sqrt{3}}^6 \frac{6}{\sqrt{36 - x^2}}\, dx$

Although this integral is undefined at $x = 0$, a graphing utility still gives $L \approx 3.142$.

25. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$

$s \approx 5$

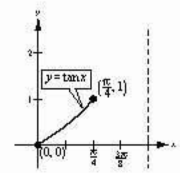
Matches (b)



26. $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$

$s \approx 1$

Matches (e)



27. $y = x^3, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$

(b) $d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2}$

≈ 64.525

(c) $s = \displaystyle\int_0^4 \sqrt{1 + (3x^2)^2}\, dx = \int_0^4 \sqrt{1 + 9x^4}\, dx \approx 64.666$ (Simpson's Rule, $n = 10$)

(d) 64.672

28. $f(x) = (x^2 - 4)^2, \quad [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

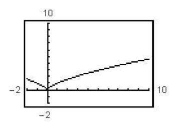
(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2}$

≈ 160.151

(c) $s = \displaystyle\int_0^4 \sqrt{1 + [4x(x^2 - 4)]^2}\, dx \approx 159.087$

(d) 160.287

29. (a) $f(x) = x^{2/3}$



(b) No, $f'(0)$ is not defined.

(c) $f'(x) = \frac{2}{3}x^{-1/3}$

$$1 + f'(x)^2 = 1 + \frac{4}{9x^{2/3}} = \frac{9x^{2/3} + 4}{9x^{2/3}}$$

Divide $[-1, 8]$ into two intervals.

$$[-1, 0]: \; s_1 = \int_{-1}^{0} \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\,dx$$
$$= \frac{-1}{3}\int_{-1}^{0} \sqrt{9x^{2/3} + 4}\,\frac{1}{x^{1/3}}\,dx, \quad (x \le 0)$$
$$= -\frac{1}{18}\int_{-1}^{0} (9x^{2/3} + 4)^{1/2}\left(\frac{6}{x^{1/3}}\right)dx$$
$$= -\frac{1}{27}(9x^{2/3} + 4)^{3/2}\Big]_{-1}^{0}$$
$$= -\frac{1}{27}(4^{3/2} - 13^{3/2})$$
$$= -\frac{1}{27}(8 - 13^{3/2}) \approx 1.4397$$

$$[0, 8]: \; s_2 = \int_{0}^{8} \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\,dx$$
$$= \frac{1}{3}\int_{0}^{8} \sqrt{9x^{2/3} + 4}\,\frac{1}{x^{1/3}}\,dx, \quad (x \ge 0)$$
$$= \frac{1}{27}(9x^{2/3} + 4)^{3/2}\Big]_{0}^{8}$$
$$= \frac{1}{27}(40^{3/2} - 4^{3/2})$$
$$= \frac{1}{27}(40^{3/2} - 8) \approx 9.0734$$

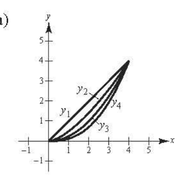
$$s_1 + s_2 = \frac{1}{27}[40^{3/2} - 8 - 8 + 13^{3/2}]$$
$$= \frac{1}{27}[40^{3/2} + 13^{3/2} - 16] \approx 10.5131$$

30. $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$
$$y = (4 - x^{2/3})^{3/2}, \quad 0 \le x \le 8$$
$$y' = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$
$$\frac{1}{4}s = \int_0^8 \sqrt{\frac{4}{x^{2/3}}}\,dx$$
$$= 2\int_0^8 x^{-2/3}\,dx = 6x^{1/3}\Big]_0^8 = 12$$

Total length: $s = 4(12) = 48$

31. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1, \; L_1 = \int_0^4 \sqrt{2}\,dx \approx 5.657$

$$y_2' = \frac{3}{4}x^{1/2}, \; L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\,dx \approx 5.759$$
$$y_3' = \frac{1}{2}x, \; L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\,dx \approx 5.916$$
$$y_4' = \frac{5}{16}x^{3/2}, \; L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\,dx \approx 6.063$$

32. Let $y = \ln x$, $1 \le x \le e$, $y' = \dfrac{1}{x}$ and $L_1 = \displaystyle\int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx$.

Equivalently, $x = e^y$, $0 \le y \le 1$, $\dfrac{dx}{dy} = e^y$, and $L_2 = \displaystyle\int_0^1 \sqrt{1 + e^{2y}}\, dy = \int_0^1 \sqrt{1 + e^{2x}}\, dx$.

Numerically, both integrals yield $L = 2.0035$.

33. $y = \dfrac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0$, $y = \frac{2}{3}$. Thus, the fleeing object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3}\left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right] = \left(\frac{1}{2}\right)\frac{x - 1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x + 1}{2x^{1/2}}\, dx = \frac{1}{2}\int_0^1 (x^{1/2} + x^{-1/2})\, dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

34.

$$y = 31 - 10(e^{x/20} + e^{-x/20})$$

$$y' = -\frac{1}{2}(e^{x/20} - e^{-x/20})$$

$$1 + (y')^2 = 1 + \frac{1}{4}(e^{x/10} - 2 + e^{-x/10})$$

$$= \left[\frac{1}{2}(e^{x/20} + e^{-x/20})\right]^2$$

$$s = \int_{-20}^{20} \sqrt{\left[\frac{1}{2}(e^{x/20} + e^{-x/20})\right]^2}\, dx$$

$$= \frac{1}{2}\int_{-20}^{20} (e^{x/20} + e^{-x/20})\, dx$$

$$= \Big[10(e^{x/20} - e^{-x/20})\Big]_{-20}^{20}$$

$$= 20\left(e - \frac{1}{e}\right) \approx 47 \text{ ft}$$

Thus, there are $100(47) = 4700$ square feet of roofing on the barn.

35.

$$y = 20 \cosh \frac{x}{20}, \quad -20 \le x \le 20$$

$$y' = \sinh \frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2 \frac{x}{20} = \cosh^2 \frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh \frac{x}{20}\, dx = 2\int_0^{20} \cosh \frac{x}{20}\, dx$$

$$= 2(20) \sinh \frac{x}{20}\Big]_0^{20} = 40 \sinh(1) \approx 47.008 \text{ m}$$

36. $y = 693.8597 - 68.7672 \cosh 0.0100333x$

$$y' = -0.6899619478 \sinh 0.0100333x$$

$$s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478 \sinh 0.0100333x)^2}\, dx$$

$$\approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

37.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\, dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= \left[3 \arcsin \frac{x}{3}\right]_0^2 = 3\left(\arcsin \frac{2}{3} - \arcsin 0\right)$$

$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

38.

$$y = \sqrt{25 - x^2}$$

$$y' = \frac{-x}{\sqrt{25 - x^2}}$$

$$1 + (y')^2 = \frac{25}{25 - x^2}$$

$$s = \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}}\, dx = \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}}\, dx$$

$$= \left[5 \arcsin \frac{x}{5}\right]_{-3}^4 = 5\left[\arcsin \frac{4}{5} - \arcsin\left(-\frac{3}{5}\right)\right]$$

$$\approx 7.8540$$

$$\frac{1}{4}[2\pi(5)] \approx 7.8540 = s$$

39. $y = \frac{x^3}{3}$

$y' = x^2, \quad [0, 3]$

$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\, dx$$
$$= \frac{\pi}{6}\int_0^3 (1 + x^4)^{1/2}(4x^3)\, dx$$
$$= \left[\frac{\pi}{9}(1 + x^4)^{3/2}\right]_0^3$$
$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

40. $y = 2\sqrt{x}$

$y' = \frac{1}{\sqrt{x}}, \quad [4, 9]$

$$S = 2\pi\int_4^9 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$$
$$= 4\pi\int_4^9 \sqrt{x + 1}\, dx$$
$$= \frac{8}{3}\pi(x + 1)^{3/2}\Big]_4^9$$
$$= \frac{8\pi}{3}(10^{3/2} - 5^{3/2}) \approx 171.258$$

41. $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$
$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \quad [1, 2]$$
$$S = 2\pi\int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$
$$= 2\pi\int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$
$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

42. $y = \frac{x}{2}$

$$y' = \frac{1}{2}$$
$$1 + (y')^2 = \frac{5}{4}, \quad [0, 6]$$
$$S = 2\pi\int_0^6 \frac{x}{2}\sqrt{\frac{5}{4}}\, dx$$
$$= \left[\frac{2\pi\sqrt{5}}{8}x^2\right]_0^6 = 9\sqrt{5}\pi$$

43. $y = \sqrt[3]{x} + 2$

$y' = \frac{1}{3x^{2/3}}, \quad [1, 8]$

$$S = 2\pi\int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx$$
$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\, dx$$
$$= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\, dx$$
$$= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8$$
$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

44. $y = 9 - x^2, \quad [0, 3]$

$y' = -2x$

$$S = 2\pi\int_0^3 x\sqrt{1 + 4x^2}\, dx$$
$$= \frac{\pi}{4}\int_0^3 (1 + 4x^2)^{1/2}(8x)\, dx$$
$$= \left[\frac{\pi}{6}(1 + 4x^2)^{3/2}\right]_0^3$$
$$= \frac{\pi}{6}(37^{3/2} - 1) \approx 117.319$$

45. $y = \sin x$

$y' = \cos x, \quad [0, \pi]$

$$S = 2\pi\int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx$$
$$\approx 14.4236$$

46. $y = \ln x$

$$y' = \frac{1}{x}$$
$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, \quad [1, e]$$
$$S = 2\pi\int_1^e x\sqrt{\frac{x^2 + 1}{x^2}}\, dx = 2\pi\int_1^e \sqrt{x^2 + 1}\, dx$$
$$\approx 22.943$$

47. A rectifiable curve is one that has a finite arc length.

48. The precalculus formula is the distance formula between two points. The representative element is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta xi.$$

49. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

50. The surface of revolution given by f_1 will be larger. $r(x)$ is larger for f_1.

51.

$$y = \frac{hx}{r}$$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi\int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\,dx$$

$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

52.

$$y = \sqrt{r^2 - x^2}$$

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$

$$S = 2\pi\int_{-r}^{r}\sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\,dx$$

$$= 2\pi\int_{-r}^{r} r\,dx = \Big[2\pi rx\Big]_{-r}^{r} = 4\pi r^2$$

53.

$$y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi\int_0^2 \frac{3x}{\sqrt{9 - x^2}}\,dx$$

$$= -3\pi\int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\,dx$$

$$= \Big[-6\pi\sqrt{9 - x^2}\Big]_0^2$$

$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 54.

54. From Exercise 53 we have:

$$S = 2\pi\int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\,dx$$

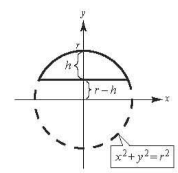
$$= -r\pi\int_0^a \frac{-2x\,dx}{\sqrt{r^2 - x^2}}$$

$$= \Big[-2r\pi\sqrt{r^2 - x^2}\Big]_0^a$$

$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$

$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$

$= 2\pi rh$ (where h is the height of the zone)



55.

$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2})\,dx$$

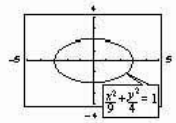
$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in.}^2$$

Amount of glass needed: $V = \frac{\pi}{27}\left(\frac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in.}^3$

56. (a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \dfrac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \dfrac{x^2}{9}}$



(b) $y = 2\sqrt{1 - \dfrac{x^2}{9}}, \quad 0 \le x \le 3$

$$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(\frac{-2x}{9}\right)$$

$$= \frac{-2x}{9\sqrt{1-(x^2/9)}} = \frac{-2x}{3\sqrt{9-x^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\,dx$$

(c) You cannot evaluate this definite integral, since the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

57. (a) We approximate the volume by summing six disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

$$V \approx \sum_{i=1}^{6} \pi r_i^2(3) = \sum_{i=1}^{6} \pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6} C_i^2$$

$$= \frac{3}{4\pi}\left[\left(\frac{50+65.5}{2}\right)^2 + \left(\frac{65.5+70}{2}\right)^2 + \left(\frac{70+66}{2}\right)^2 + \left(\frac{66+58}{2}\right)^2 + \left(\frac{58+51}{2}\right)^2 + \left(\frac{51+48}{2}\right)^2\right]$$

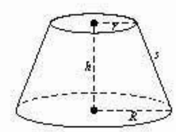
$$= \frac{3}{4\pi}\left[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2\right]$$

$$= \frac{3}{4\pi}[21813.625] = 5207.62 \text{ cubic inches}$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum:

$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$

$$= \left(\frac{50 + 65.5}{2}\right)\left[9 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}.$$



Adding the six frustums together:

$$S \approx \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5+70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2} +$$

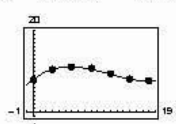
$$\left(\frac{70+66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66+58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2} +$$

$$\left(\frac{58+51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51+48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$

$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37$$

$$= 1168.64$$

(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



(d) $V = \displaystyle\int_0^{18} \pi r^2\,dy \approx 5275.9$ cubic inches

$$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\,dy$$

≈ 1179.5 square inches

58. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

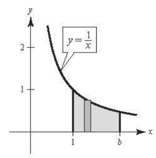
(b) Area $= \int_0^{400} f(x)\,dx \approx 131{,}734.5$ square feet ≈ 3.0 acres (1 acre = 43,560 square feet)

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2}\,dx \approx 794.9$ feet

(Answers will vary.)

59. (a) $V = \pi \int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$



(b) $S = 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$

$= 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$

$= 2\pi \int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Since

$\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0$ on $[1, b]$

we have

$\int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \left[\ln x\right]_1^b = \ln b$

and $\lim_{b\to\infty} \ln b \to \infty$. Thus,

$\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4+1}}{x^3}\,dx = \infty.$

60. (a) Area of circle with radius L: $A = \pi L^2$

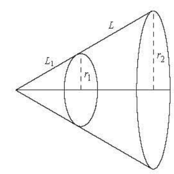
Area of sector with central angle θ (in radians):

$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2}L^2\theta$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL.$

(c) The lateral surface area of the frustum is the difference of the large cone and the small one.



$S = \pi r_2(L + L_1) - \pi r_1 L_1$

$= \pi r_2 L + \pi L_1(r_2 - r_1)$

By similar triangles, $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1)$. Hence,

$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$

$= \pi L(r_1 + r_2).$

61. Individual project

62. Essay

63. $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$

$$y = (4 - x^{2/3})^{3/2}, \quad 0 \le x \le 8$$

$$y' = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$

$$S = 2\pi\int_0^8 (4 - x^{2/3})^{3/2}\sqrt{\frac{4}{x^{2/3}}}\,dx$$

$$= 4\pi\int_0^8 \frac{(4 - x^{2/3})^{3/2}}{x^{1/3}}\,dx$$

$$= \left[-\frac{12\pi}{5}(4 - x^{2/3})^{5/2}\right]_0^8 = \frac{192\pi}{5}$$

[Surface area of portion above the x-axis]

64. $y^2 = \frac{1}{12}x(4 - x)^2, \quad 0 \le x \le 4$

$$y = \frac{(4 - x)\sqrt{x}}{\sqrt{12}}$$

$$y' = \frac{(4 - 3x)\sqrt{3}}{12\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{(4 - 3x)^2}{48x}$$

$$= \frac{48x + 16 - 24x + 9x^2}{48x} = \frac{(4 + 3x)^2}{48x}, \quad x \ne 0$$

$$S = 2\pi\int_0^4 \frac{(4 - x)\sqrt{x}}{\sqrt{12}} \cdot \frac{(4 + 3x)}{\sqrt{48x}}\,dx$$

$$= 2\pi\int_0^4 \frac{(4 - x)(4 + 3x)}{24}\,dx$$

$$= \frac{\pi}{12}\int_0^4 (16 + 8x - 3x^2)\,dx$$

$$= \frac{\pi}{12}\left[16x + 4x^2 - x^3\right]_0^4$$

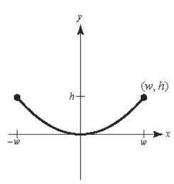
$$= \frac{\pi}{12}[64 + 64 - 64] = \frac{16\pi}{3}$$

65. $y = kx^2$, $y' = 2kx$

$$1 + (y')^2 = 1 + 4k^2x^2$$

$$h = kw^2 \implies k = \frac{h}{w^2} \implies 1 + (y') = 1 + \frac{4h^2}{w^4}x^2$$

By symmetry, $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx.$



66. $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx$

$$= 2\int_0^{700} \sqrt{1 + \frac{4(155)^2x^2}{700^4}}\,dx$$

$\approx$ 1444.5 meters

67. Let (x_0, y_0) be the point on the graph of $y^2 = x^3$ where the tangent line makes an angle of 45° with the x-axis.

$$y = x^{3/2}$$

$$y' = \tfrac{3}{2}x^{1/2} = 1$$

$$x_0 = \tfrac{4}{9}$$

$$L = \int_0^{4/9} \sqrt{1 + \tfrac{9}{4}x}\,dx = \tfrac{8}{27}\left(2\sqrt{2} - 1\right)$$