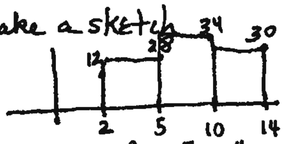
Riemann Sums and Trapezoidal Approximations

Answers

① Make a sketch

(D)



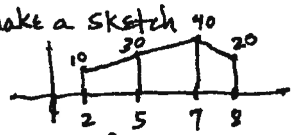
$$\int_2^{14} f(x)\,dx = 3(28) + 5(34) + 4(30) = 84 + 170 + 120 = \underline{374}$$

using right end pts (start at 14) and draw to the left. Start at (10, 34) and draw to the left, etc.

3 5 4

② Make a sketch

(C)

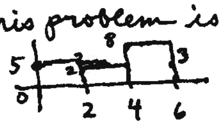


Now draw traps. $\int_2^8 f(x)\,dx = \frac{1}{2}(3)(10+30) + \frac{1}{2}(2)(30+40) + \frac{1}{2}(1)(40+20) =$

$60 + 70 + 30 = \underline{160}$

③ This problem is more than just using the table. Make a sketch.

(E)



given $v(0) = 11$, find $v(6)$ using left end pts.

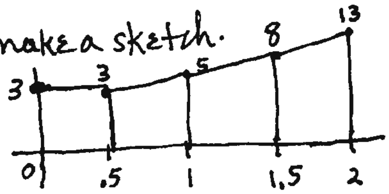
$\int_0^6 a(t)\,dt = v(6) - v(0) \Rightarrow v(6) = v(0) + \int_0^6 a(t)\,dt$

from given — from sketch

$11 + 2(5) + 2(2) + 2(8) = 11 + 10 + 4 + 16 = 41$

④ Make a sketch.

(B)



draw traps.

$$\int_0^2 f(x)\,dx = \frac{1}{2}\left(\frac{1}{2}\right)(3+3) + \frac{1}{2}\left(\frac{1}{2}\right)(3+5) + \frac{1}{2}\left(\frac{1}{2}\right)(5+8) + \frac{1}{2}\left(\frac{1}{2}\right)(8+13)$$

$$= \frac{1}{4}(6) + \frac{1}{4}(8) + \frac{1}{4}(13) + \frac{1}{4}(21)$$

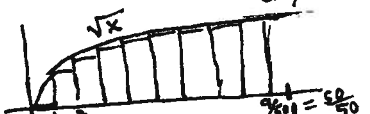
$$= 1.5 + 2 + 3.25 + 5.25 = \boxed{12}$$

⑤ This is one I would probably guess, but if I had time I would start with the easiest choice and work from there. Easiest choice is B. Now let's analyze it. If $n = 50$, I would have $\frac{1}{50}$ in the front. (so far so good) to represent the width of each of the 50 rect. from 0 to 1. The height of each rect would be $f\left(\frac{x}{50}\right)$ for 50 rect $n = \frac{1-0}{50}$

(B)

$\frac{1}{50}\left(\sqrt{\frac{0}{50}} + \sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \dots \sqrt{\frac{50}{50}}\right)$

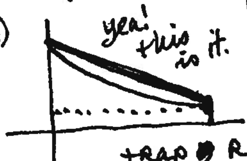
B works, yea, cause I'd be done with this



⑥ Just draw ONE trap and ONE right end pt. Rect on each choice to find where trap is over and rect is under. Dots are rect. Dark lines are traps.

(A)

A) yea! this is it. but, I'll demo the others.

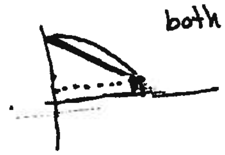


trap over, Rect under

B) trap is really a Δ.

rect over, trap under

C) both the trap & rect are under app.



D)

Rect over, trap over

E)

Rect over, trap (really Δ) actual area

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Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	$2: \begin{cases} 1: \text{estimate} \\ 1: \text{conclusion with reason} \end{cases}$
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\vert_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$	$3: \begin{cases} 2: \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	$2: \begin{cases} 1: \text{approximation} \\ 1: \text{explanation} \end{cases}$
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

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Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

(a) Midpoint Riemann sum is
$10 \cdot [v(5) + v(15) + v(25) + v(35)]$
$= 10 \cdot [9.2 + 7.0 + 2.4 + 4.3] = 229$
The integral gives the total distance in miles that the plane flies during the 40 minutes.

3 : { 1 : $v(5) + v(15) + v(25) + v(35)$; 1 : answer; 1 : meaning with units }

(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.

2 : { 1 : two instances; 1 : justification }

(c) $f'(23) = -0.407$ or -0.408 miles per minute^2

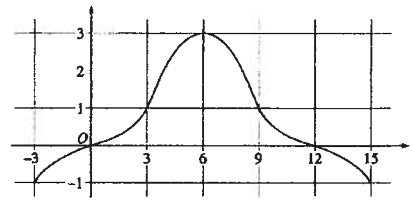
1 : answer with units

(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$
$= 5.916$ miles per minute

3 : { 1 : limits; 1 : integrand; 1 : answer }

Question 4

The graph of a differentiable function f on the closed interval $[-3,15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.



Graph of f

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

(a) $g(6) = 5 + \int_6^6 f(t)\,dt = 5$

$g'(6) = f(6) = 3$

$g''(6) = f'(6) = 0$

3 { 1 : $g(6)$; 1 : $g'(6)$; 1 : $g''(6)$ }

(b) g is decreasing on $[-3,0]$ and $[12,15]$ since $g'(x) = f(x) < 0$ for $x < 0$ and $x > 12$.

3 { 1 : $[-3,0]$; 1 : $[12,15]$; 1 : justification }

(c) The graph of g is concave down on $(6,15)$ since $g' = f$ is decreasing on this interval.

2 { 1 : interval; 1 : justification }

(d) $\frac{3}{2}(-1 + 2(0 + 1 + 3 + 1 + 0) - 1)$
$= 12$

1 : trapezoidal method

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Question 2

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ °C/day}$$

2 : { 1 : difference quotient; 1 : answer (with units) }

(b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1$ °C

2 : { 1 : trapezoidal method; 1 : answer }

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549$ °C/day

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

2 : { 1 : $P'(12)$ (with or without units); 1 : interpretation }

(d) $\frac{1}{15}\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757$ °C

3 : { 1 : integrand; 1 : limits and average value constant; 1 : answer }

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3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24–hour period.
   (a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.
   (b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.
   (c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24–hour time period. Indicate units of measure.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

Solution	Scoring
(a) $\int_0^{24} R(t)\,dt \approx 6[R(3) + R(9) + R(15) + R(21)]$ $= 6[10.4 + 11.2 + 11.3 + 10.2]$ $= 258.6$ gallons This is an approximation to the total flow in gallons of water from the pipe in the 24–hour period.	3: { 1: $R(3) + R(9) + R(15) + R(21)$; 1: answer; 1: explanation }
(b) Yes; Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.	2: { 1: answer; 1: MVT or equivalent }
(c) Average rate of flow $\approx$ average value of $Q(t)$ $= \frac{1}{24}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt$ $= 10.785$ gal/hr or 10.784 gal/hr	3: { 1: limits and average value constant; 1: $Q(t)$ as integrand; 1: answer }
(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.	1: units